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Some results on $(1, f)$-odd factors

M. Kano and Haruhide Matsuda
Department of Computer and Information Sciences
Ibaraki Unversity
Hitachi 316 Japan

Abstract

Let G be a graph and $f : V(G) \to \{1, 3, 5, \ldots\}$. Then a spanning subgraph F of G is called a $(1, f)$-odd factor if $\deg_F(x) \in \{1, 3, \ldots, f(x)\}$ for all $x \in V(G)$. We give some results on $(1, f)$-odd factors and k-critical graphs with respect to $(1, f)$-odd factor.

We consider finite graphs without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We write $|G|$ and $||G||$ for the order and the size of G, respectively. For a vertex v of G, we denote by $\deg_G(v)$ the degree of v in G, and by $N_G(v)$ the neighborhood of v. A graph G is called an r-regular graph if $\deg_G(x) = r$ for all $x \in V(G)$. An edge joining a vertex x to a vertex y is denoted by xy or yx. For a subset $S \subset V(G)$, we define $N_G(S) := \bigcup_{x \in S} N_G(x)$, and denote by $\langle S \rangle_G$ the subgraph of G induced by S, and write $G - S$ for the subgraph $\langle V(G) \setminus S \rangle_G$. We denote by $o(G)$ the number of odd components of G, where an *odd component* is a component of odd order.

We define a function f by $f : V(G) \to \{1, 3, 5, \ldots\}$, and f always denotes this function throughout this paper. Then a subgraph H of G is called a $(1, f)$-*odd subgraph* if $\deg_H(x) \in \{1, 3, \cdots, f(x)\}$ for all $x \in V(H)$, and a spanning $(1, f)$-odd subgraph is called a $(1, f)$-*odd factor*. So if $f(x) = 1$ for all $x \in V(G)$, then a $(1, f)$-odd subgraph is a matching and a $(1, f)$-odd factor is a 1-factor. The complete graph

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of order n is denoted by K_n, and the graph consisting of k disjoint copies of K_n is denoted by kK_n. A *cutset* is a minimal set of edges in a connected graph whose removal disconnects the graph. A cutset with m edges is briefly called a *m-cutset*.

Theorem A ([1]) *A graph G has a $(1,f)$-odd factor if and only if*

$$o(G-S) \le \sum_{x\in S} f(x) \quad \text{for all} \quad S \subset V(G).$$

In particular, if G is of even order and has no $(1,f)$-odd factor, then there exists a subset S_0 of $V(G)$ such that $o(G-S_0) \ge \sum_{x\in S_0} f(x)+2$.

Theorem 1 *Let G be a connected r-regular graph of even order. Suppose that for every cutset $C=\{x_1y_1, x_2y_2, \ldots, x_ty_t\}$, $x_i, y_i \in V(G)$, of G such that $G-C$ has an odd component containing all y_i's, we have $f(x_1)+f(x_2)+\cdots+f(x_t) \ge r$. Then G has a $(1,f)$-odd factor.*

Proof Let S be a nonempty subset of $V(G)$, and $D_1, \ldots, D_m$ be the odd components of $G-S$. For each D_i, $E_G(S,D_i) := \{xy \mid x \in S \text{ and } y \in V(D_i)\}$ is a cutset of G. Then by the assumption, we have

$$rm \le \sum_{i=1}^{m} \sum_{xy\in E_G(S,D_i)} f(x) \;\le\; \sum_{x\in S} \deg_G(x) f(x) = \sum_{x\in S} r f(x).$$

Therefore $o(G-S) = m \le \sum_{x\in S} f(x)$, and the theorem follows from Theorem A. □

Corollary 2 *(i) Let G be a connected 3-regular graph and define f as*

$$f(x) = \begin{cases} 3 & \text{if } x \text{ is a cut-vertex,} \\ 1 & \text{otherwise.} \end{cases}$$

Then G has a $(1,f)$-odd factor.

(ii) Let G an n-edge-connected r-regular graph of even order. Let m be an integer such that $m \in \{n,\ n+1\}$ and $m \equiv r \pmod 2$. If we define f by $f(x)$:=the least odd integer greater than or equal to r/m, for all $x \in V(G)$, then G has a $(1,f)$-odd factor.

Proof We prove only (ii) since (i) can be proved similarly. Let $C = \{x_1y_1, \ldots, x_ty_t\}$ be a cutset of G, and D an odd component of $G - C$ containing all y_i's. Then $\sum_{x \in V(D)} \deg_G(x) = 2||D|| + t \equiv t \pmod 2$. Combined with $\sum_{x \in V(D)} \deg_G(x) = r|D| \equiv r \pmod 2$ we obtain $t \equiv r \pmod 2$. Thus $t \geq m$, which implies $f(x_1) + \cdots + f(x_t) \geq t \times r/m \geq r$. Therefore G has a $(1, f)$-odd factor by Theorem 1. □

In the second statement (ii) we obtain for $n = 2,\ r = 3$ the following well-known theorem by Petersen (Theorem 3.4.1 [3]): Every 2-edge-connected 3-regular graph has a 1-factor. Moreover the first statement (i) is also a generalization of this theorem since a 2-edge-connected 3-regular graph is 2-connected and has no cut-vertex.

Let $k \geq 1$ be an integer. Then a graph G is said to be *k-extendable* if for every matching M with k edges, G has a 1-factor containing M. Similarly G is said to be *k-critical* if for every subset $X \subseteq V(G)$ of k vertices, $G - X$ has a 1-factor. Let $\mathcal{P}$ be a set of graphs with a given constant order. Then we say that G is *$\mathcal{P}$-critical with respect to $(1, f)$-odd factor* if for every subgraph H of G isomorphic to a graph in $\mathcal{P}$, $G - V(H)$ has a $(1, f)$-odd factor. Then G is $\{kK_2\}$-critical with respect to 1-factor if and only if G is $2k$-extendable. Similarly G is $\{kK_1\}$-critical with respect to 1-factor if and only if G is k-critical.

Theorem B (Yu [5] and Chen [2]) *Let $\mathcal{P}$ be a set of graphs of order k. Then G is $\mathcal{P}$-critical with respect to 1-factor if and only if*

$$o(G - S) \leq |S| - k,$$

for all subset $S \subseteq V(G)$ such that $\langle S \rangle_G$ has a subgraph in $\mathcal{P}$. In particular, G is k-critical if and only if $o(G - S) \leq |S| - k$ for all $S \subseteq V(G)$, $|S| \geq k$. Moreover, for an even integer k, G is $k/2$-extendable if and only if $o(G - S) \leq |S| - k$ for all $S \subseteq V(G)$ such that $\langle S \rangle_G$ has a matching of $k/2$ edges.

We generalize the above theorem as follows.

Theorem 3 *Let $k \geq 1$ be an integer, G a graph of order at least $k+2$, and $\mathcal{P}$ a set of graphs of order k. Then G is $\mathcal{P}$-critical with respect to $(1, f)$-factor if and only if for all subset $S \subseteq V(G)$ such that $\langle S \rangle_G$ has a subgraph isomorphic to a graph in $\mathcal{P}$, we have*

$$o(G - S) \leq \sum_{x \in S} f(x) - f_S(k) \qquad (1)$$

where $f_S(k) := \max\{\sum_{x\in X} f(x) \mid X \subseteq S,\ |X| = k, \langle X\rangle_G$ *has a subgraph in* $\mathcal{P}\}$.

Proof Assume that G is $\mathcal{P}$-critical. Let $S \subseteq V(G)$ be such that $\langle S\rangle_G$ has a subgraph in $\mathcal{P}$. Take a subset $Y \subseteq S$ such that $|Y| = k$ and $\sum_{x\in Y} f(x) = f_S(k)$. Then $G - Y$ has a $(1, f)$-odd factor, and thus for $S' := S \setminus Y$ we have by Theorem 1,

$$\begin{aligned} o(G-S) &= o((G-Y)-S')) \le \sum_{x\in S'} f(x) \\ &= \sum_{x\in S} f(x) - \sum_{x\in Y} f(x) = \sum_{x\in S} f(x) - f_S(k), \end{aligned}$$

which implies (1).

Conversely we assume that (1) holds. Let $H \in \mathcal{P}$ be a subgraph of G, and put $X = V(H)$. For any subset $S' \subseteq V(G-X)$, let $S := S' \cup X$. Then

$$\begin{aligned} o((G-X)-S') &= o(G-S) \le \sum_{x\in S} f(x) - \max\{f_S(k)\} \\ &\le \sum_{x\in S} f(x) - \sum_{x\in X} f(x) = \sum_{x\in S'} f(x). \end{aligned}$$

Therefore $G - X$ has a $(1, f)$-odd factor by Theorem A. □

For an integer $k \ge 1$ and a graph G with $|G| \ge k + 2$, we say that G is *k-critical with respect to $(1, f)$-odd factor* if for every subset $X \subset V(G)$ with $|X| = k$, $G - X$ has a $(1, f)$-odd factor. The following corollary is an easy consequence of Theorem 3.

Corollary 4 *Let $k \ge 1$ be an integer, G a graph of order at least $k+2$. Then G is k-critical with respect to $(1, f)$-odd factor if and only if*

$$o(G-S) \le \sum_{x\in S} f(x) - \max\{\sum_{x\in X} f(x) \mid X \subseteq S,\ |X| = k\} \qquad (2)$$

for all subsets $S \subseteq V(G)$ with $|S| \ge k$.

Theorem 5 *Let $k \ge 1$ be an integer and G a graph with $|G| \ge k + 2$ and $|G| \equiv k \pmod 2$. If G is k-critical with respect to $(1, f)$-odd factor, then (i) for every integer m such that $0 \le m \le k$ and $m \equiv k \pmod 2$, G is m-critical with respect to $(1, f)$-odd factor; and (ii) G is k-connected.*

Proof (i) It suffices to prove that G is $(k-2)$-critical. Let us define $f_S(k) := \max\{\sum_{x \in X} f(x) \mid X \subseteq S,\ |X| = k\}$. Let $S \subset V(G)$ with $|S| \geq k-2$. We shall show that the following inequality holds, which implies that G is $(k-2)$-critical by Corollary 4.

$$o(G-S) \leq \sum_{x \in S} f(x) - f_S(k-2). \tag{3}$$

If $|S| \geq k$, then $o(G-S) \leq \sum_{x\in S} f(x) - f_S(k) \leq \sum_{x \in S} f(x) - f_S(k-2)$. Hence we may assume $|S| \leq k-1$. It is clear that we may assume $o(G-S) > 0$. If $G-S$ has an even component C, then for subset $T \subset V(C)$ of $k-|S|$ vertices, $G-(S \cup T)$ has no $(1, f)$-odd factor since it has an odd component, which contradicts the assumption that G is k-critical. Thus we may assume that every component of $G-S$ is odd. If $|S| = k-1$, then we take a vertex v from an odd component D of $G-S$ such that $D-v$ is connected. Then since G is k-critical, $o(G-S) = o(G-(S \cup \{v\})) + 1 \leq \sum_{x \in S \cup \{v\}} f(x) - f_{S\cup\{v\}}(k) + 1 = 1 \leq \sum_{x\in S} f(x) - f_S(k-2)$. If $|S| = k-2$ and if one of odd components of $G-S$, say D, has order at least three, then we take two vertices v_1 and v_2 from D such that $D - \{v_1, v_2\}$ is connected. Then $o(G-S) = o(G-(S\cup\{v_1, v_2\})) \leq \sum_{x \in S\cup\{v_1,v_2\}} f(x) - f_{S\cup\{v_1,v_2\}}(k) = 0 = \sum_{x\in S} f(x) - f_S(k-2)$. If $|S| = k-2$ and if every odd component of $G-S$ is an isolated vertex, then $G-S$ has at least four isolated vertices as $|G| \geq k+2$. Thus if we take two isolated vertices v_1 and v_2 of $G-S$, then $G-(S\cup\{v_1,v_2\})$ has no $(1, f)$-odd factor, a contradiction. Consequently (3) holds.

(ii) We prove (ii) by induction on $|G|$ and k. If $k = 1$, then we can easily see that G is connected. Hence we may assume $k \geq 2$. Assume that $|G| = k+2$ and G is not k-connected. Then there exists $A \subset V(G)$ such that $|A| = k-1$ and $G-A$ is disconnected. Since one component of $G-A$, say w, is an isolated vertex, for any vertex $v \in V(G-A) \setminus \{w\}$, $G-(A\cup\{v\})$ has no $(1, f)$-odd factor, a contradiction. Thus G is k-connected.

Suppose $|G| \geq k+3$. Let x be any vertex of G. Then clearly $G - \{x\}$ is $(k-1)$-critical as G is k-critical. So $G - \{x\}$ is $(k-1)$-connected by the induction hypothesis, which implies that G is k-connected. □

Theorem C (Topp and Vestergaard [4]) *Let G be an n-connected graph of even order. If no vertex v of G is the center of an induced star $K(1, nf(v)+1)$, then G has a $(1, f)$-odd factor.*

Theorem 6 *Let k, m, n be positive integers such that $n \geq m$, and G an n-connected graph with $|G| \geq k+2$ and $|G| \equiv k \pmod 2$. Put $f_G(k) := \max\{\sum_{x \in X} f(x) \mid X \subset V(G),\ |X| = k\}$. If no vertex v of G is the center of an induced star $K(1, mf(v)+1)$ and $m\, f_G(k) < 2n$, then G is k-critical with respect to $(1,f)$-odd factor.*

Proof We use $f_S(k)$ defined in the Proof of Theorem 5. Suppose that G satisfies the conditions in the above theorem but is not k-critical. Then by Corollary 4 and since the parity of $o(G-S)$ equals to that of $\sum_{x \in S} f(x) - f_S(k)$, there exists a subset $S \subset V(G)$ with $|S| \geq k$ such that

$$o(G-S) \geq \sum_{x \in S} f(x) - f_S(k) + 2. \tag{4}$$

Let $D_1, \ldots, D_r$ be the odd components of $G-S$, where $r = o(G-S)$. We construct a bipartite graph B with partite sets S and $\{D_i \mid 1 \leq i \leq r\}$ as follows: A vertex $x \in S$ and D_i are joined by an edge of B if and only if G has an edge joining x and a vertex of D_i.

Since G is n-connected, we have $\deg_B(D_i) \geq n$ for all i. Since no vertex v of G is the center of an induced star $K(1, mf(v)+1)$, we have $\deg_B(x) \leq mf(x)$ for all $x \in S$. Therefore by counting the number of edges of B joining S and $\{D_i \mid 1 \leq r \leq r\}$, we obtain

$$n \cdot o(G-S) \leq \sum_{i=1}^{r} \deg_B(D_i) = \sum_{x \in S} \deg_B(x) \leq \sum_{x \in S} mf(x). \tag{5}$$

Let $Y \subseteq S$ such that $|Y| = k$ and $\sum_{x \in Y} f(x) = f_S(k)$. By (4) and (5), we have

$$\sum_{x \in S} f(x) - f_S(k) + 2 \leq \frac{m}{n} \sum_{x \in S} f(x).$$

$$(1 - \frac{m}{n}) \sum_{x \in S} f(x) - \sum_{x \in Y} f(x) \leq -2.$$

$$-\frac{m}{n} \sum_{x \in Y} f(x) \leq -2. \tag{6}$$

On the other hand, we have $(m/n) \sum_{x \in Y} f(x) \leq (m/n) f_G(k) < 2$. This contradicts (6), and thus the proof is complete. □

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