STAR PARTITIONS OF GRAPHS

Y. Egawa
Department of Applied Mathematics
Science University of Tokyo
Kagurazaka, Shinjuku Tokyo 162 Japan

M. Kano
Department of Computer and Information Sciences
Ibaraki University
Hitachi 316 Japan

Alexander K. Kelmans
RUTCOR, Rutgers University, New Brunswick, NJ 08903
and
Department of Mathematics, University of Puerto Rico, San Juan, PR 00931
USA

JGT # 2825

Mailing address
M. Kano
Department of Computer and Information Sciences
Ibaraki University
Hitachi 316 Japan

Abstract

Let G be a graph and $n \geq 2$ an integer. We prove that the following are equivalent: (i) there is a partition $(V_1, \ldots, V_m)$ of $V(G)$ such that each V_i induces one of stars $K_{1,1}, \ldots, K_{1,n}$, and (ii) for every subset S of $V(G)$, $G \backslash S$ has at most $n|S|$ components with the property that each of their blocks is an odd order complete graph.

1 Introduction

We consider finite undirected graphs without loops or multiple edges. The notion and facts on graphs that are used but not described here can be found in [2]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$.

Given a graph G and a set $\mathcal{F}$ of graphs, an *$\mathcal{F}$–factor* of G is a spanning subgraph F of G such that every component of F is a member of $\mathcal{F}$.

Different types of $\mathcal{F}$–factor problems for a graph have extensively been studied by many authors for different classes of families $\mathcal{F}$ (e.g. [1, 3, 5, 6, 7, 8, 9, 10]).

For a vertex subset S of G, let $G - S$ denote the graph obtained from G by deleting the vertices in S together with their incident edges. We write $G - v$ for $G - \{v\}$.

A vertex x of a connected graph G is called a *cut–vertex* if $G - x$ is disconnected. A maximal connected subgraph of G without cut–vertices is called a *block of* G. A block of G is said to be *odd* or *even* according to whether it has an odd or an even number of vertices. A graph G is called a *cactus* if G is connected and each of block of G is a complete graph. A cactus is called an *odd–cactus* if each of its blocks is odd. Note that the graph with one vertex is an odd–cactus. A *star* is a graph having a vertex incident to each of its edges. The *n–star* is a star with n edges, i.e. the n–star is the complete bipartite graph $K_{1,n}$. Let $\mathcal{S}_n$ denote the set of all stars with at least one and at most n edges. Let $\mathcal{S}$ denote the set of all stars with at least one edge, i.e. $S = S_\infty$.

In this paper we consider an $\mathcal{S}_n$–factor A of G with an additional property, i.e.

$(p1)$ every component of A is isomorphic to a star of at least one and at most n edges, and moreover

$(p2)$ every component of A is an induced subgraph of G.

We call such a factor a *strong $\mathcal{S}_n$–factor* of G.

Let $oc(G)$ denote the number of those components of G that are odd–cacti.

Theorem A (Kelmans [7], Saito and Watanabe[10]) *A connected graph G has a strong $\mathcal{S}$–factor if and only if G is not an odd–cactus.*

In this paper we prove the following theorem:

Theorem 1 *Let $n \geq 2$ be an integer. Then a graph G has a strong $\mathcal{S}_n$–factor if and only if*

$$oc(G-S) \leq n|S| \qquad \text{for all subsets} \qquad S \subset V(G). \tag{1}$$

Clearly a strong $\mathcal{S}_1$–factor is a 1–factor. Therefore if $n = 1$ then the above theorem does not hold by Tutte's 1–factor theorem. It is easy to see that an odd–cactus has no strong $\mathcal{S}$–factor (see Lemma 2). Moreover, if $n \geq |V(G)|$ then G satisfies (1) if and only if no component of G is an odd–cactus. Therefore the above theorem implies Theorem A.

An algorithmic proof of Theorem 1 is given in [7].

2 Preliminary results

Our notation is standard [2] except possibly for the following. Let G be a graph.

$N_G(v)$ is the set of vertices of G adjacent to a vertex v,

$N_G(X)$ is the set of vertices adjacent to at least one vertex of $X \subseteq V(G)$, and

xy or yx is the edge joining vertices x and y.

Our proof is based on the following theorem which is a variant of the Matching Theorem of Hall (see [2]), and can be easily verified by a modification of most of the known proofs of the Mathching Theorem.

Theorem B (Folkman and Fulkerson [4]) *Let H be a bipartite graph with bipartition (S,T), and $n \geq 1$ be an integer. Then the following are equivalent:*

(a) $|N_H(X)| \geq |X|$ and $n|N_H(Y)| \geq |Y|$ for all $X \subseteq S$ and $Y \subseteq T$, and

(b) G has a strong $\mathcal{S}_n$–factor in which every vertex of T is of degree 1.

In order to prove our theorem we need the following simple lemma:

Lemma 2 *(i) A cactus with exactly one even block has a 1–factor. (ii) A cactus with at least one and at most two even blocks has a strong $\mathcal{S}_2$–factor. (iii) If G is an odd–cactus then $G-v$ has a 1–factor for every vertex v of G. (iv) An odd–cactus does not have a strong $\mathcal{S}$–factor.*

Proof We prove (i) by induction on $|V(G)|$. Let B be the even block of G. Then B clearly has a 1–factor. Thus if $G = B$, then the proof is complete. Therefore we may assume that $G \neq B$. Clearly every component of $G - V(B)$ is a cactus with exactly one even block. Therefore by the induction hypothesis, they have 1–factors, and hence G also has a 1–factor.

We prove (ii). By (i), we can assume that G has exactly two even blocks, say B_1 and B_2. Clearly there exists a cut–vertex x of G such that $B_1 - x$ and $B_2 - x$ are in different components of $G - x$, say in C_1 and C_2, respectively. If $x \in V(B_i)$ then let x_i be an arbitrary vertex of $B_i - x$ (so $xx_i \in E(G)$). If $x \notin V(B_i)$ then let x_i be an arbitrary vertex of $C_i - B_i$ adjacent to x. In the case $x \notin V(B_i)$ such a vertex x_i exists because the (unique) block of G containing x and a vertex in C_i is an odd clique. Then $\{x, x_1, x_2\}$ induces a 2–star S. Clearly every component of $G - S$ is a cactus having exactly one even block. By (i), every component has a 1–factor, and so G has a strong $\mathcal{S}_2$–factor.

We now prove (iii). Since all the components of $G - v$ satisfy the assumptions of (i), it follows that $G - v$ has a 1–factor.

We prove (iv) by induction on $|V(G)|$. If G consists of exactly one block then the statement is obvious. Therefore we can assume that G is not a complete graph. Then G has a cut–vertex v. For every component C of $G - v$, the subgraph of G induced by $V(C) \cup v$ is an odd–cactus and has no strong $\mathcal{S}$–factor, by the induction hypothesis. On the other hand, if G has a strong $\mathcal{S}$–factor, then for at least one component D of $G - v$, the subgraph induced by $V(D) \cup v$ must have a strong $\mathcal{S}$–factor, a contradiction. □

Remark 1 *It is easy to prove that the statements* (i) *and* (ii) *of Lemma 2 can be generalized as follows: A cactus with at least one and at most* n *even blocks has a strong* $\mathcal{S}_n$*–factor.*

3 Proof of the theorem

We first prove the necessity. Suppose that G has a vertex subset S such that $oc(G - S) > n|S|$. Assume on the contrary that G has a strong S_n–factor F. Since every vertex of S has degree at most n in F, clearly $G - S$ has a component D that is an odd cactus and that has no common vertex with any star of F having a vertex in S. Therefore $D \cap F$ is a strong S_n–factor of D. This contradicts (iv) of Lemma 2.

We next prove the sufficiency by induction on $|V(G)|$. Obviously we can assume that $|V(G)| \geq 3$. If G is not connected, then each component of G has a strong $\mathcal{S}_n$–factor, by the induction hypothesis, and therefore G also has a strong $\mathcal{S}_n$–factor. Hence we may assume that G is connected. By taking $S = \emptyset$, it follows that G is not an odd–cactus. Assume that G has two adjacent vertices x and y such that

$$oc(G - S) \leq n|S| - 2n \quad \text{for all subsets} \quad S \quad \text{with} \quad \{x, y\} \subseteq S \subset V(G).$$

Then for every $T \subset V(G - \{x, y\})$,

$$oc(G - \{x, y\} - T) = oc(G - (T \cup \{x, y\})) \leq n|T \cup \{x, y\}| - 2n = n|T|.$$

Hence by the induction hypothesis, $G - \{x, y\}$ has a strong $\mathcal{S}_n$–factor, and thus G also has a strong $\mathcal{S}_n$–factor. Therefore we may assume that for every pair of adjacent vertices x and y of G, there exists a subset $T_{xy} \subset V(G)$ such that

$$\{x, y\} \subseteq T_{xy} \quad \text{and} \quad oc(G - T_{xy}) \geq n|T_{xy}| - 2n + 1. \tag{2}$$

Let

$$m := \min\{n|X| - oc(G - X) \mid X \subset V(G) \quad \text{and} \quad oc(G - X) > 0\}.$$

Since $|T_{xy}| \geq 2$ and $n \geq 1$ we have from (2): $oc(G - T_{xy}) \geq 1$. Therefore m is well–defined. By (1) and (2), we have $0 \leq m \leq 2n - 1$, and it immediately follows from the definition of m that

$$oc(G - X) \leq n|X| - m$$

for all $X \subset V(G)$ with $oc(G - X) > 0$.

A proper subset S of $V(G)$ is said to be *tight* if $n|S| - oc(G - S) = m$ and $oc(G - S) > 0$. Tight sets will play an essential role in our proof. First we prove the following

Fact 1 *Let S be a tight set of G, and let D be a component of $G - S$ which is not an odd–cactus. Then D has a strong $\mathcal{S}_n$–factor.*

Indeed, for every $Z \subset V(D)$, we obtain

$$n|S \cup Z| - m \geq oc(G - (S \cup Z)) = oc(G - S) + oc(D - Z) = n|S| - m + oc(D - Z)$$

which means that $oc(D - Z) \leq n|Z|$ because $S \cap Z = \emptyset$. Since $|V(D)| < |V(G)|$, the component D has a strong $\mathcal{S}_n$–factor, by the induction hypothesis. Hence Fact 1 is proved.

We consider two cases.

Case 1. $oc(G - S) \geq |S|$ for some tight set S of G.

We define a bipartite graph H with bipartition (S, T) as follows: T is the set of components of $G - S$ that are odd–cacti, and vertices $x \in S$ and $C \in T$ are adjacent in H if and only if x is adjacent to a vertex in the component C of $G - S$ in G.

We first prove that

$$|N_H(X)| \geq |X| \quad \text{and} \quad n|N_H(Y)| \geq |Y| \tag{3}$$

for all $X \subseteq S$ and $Y \subseteq T$.

If $X = \emptyset$ then clearly $|N_H(X)| \geq |X|$. Suppose that $X \neq \emptyset$ and $|N_H(X)| < |X|$. Then

$$oc(G - (S \setminus X)) \geq oc(G - S) - |N_H(X)| > n|S| - m - |X| \geq n|S \setminus X| - m$$

because $n \geq 1$. So we have $n|S \setminus X| - oc(G - (S \setminus X)) < m$. This contradicts the definition of m because by $oc(G - S) \geq |S|$, we get $oc(G - (S \setminus X)) \geq oc(G - S) - |N_H(X)| > |S| - |X| \geq 0$. Thus $|N_H(X)| \geq |X|$ for every subset X of S.

Let $Y \subseteq T$. Then by the assumption of the theorem, $n|N_H(Y)| \geq oc(G - N_H(Y)) \geq |Y|$. Hence (3) holds.

Therefore by Theorem B, the graph H has a strong $\mathcal{S}_n$–factor F in which every vertex of T is of degree 1. Consequently, by Lemma 2 and Fact 1, we can obtain a strong $\mathcal{S}_n$–factor of G by making use of F.

Case 2. $oc(G - S) < |S|$ for every tight set S of G.

Let S be a tight set of G.

We shall first prove that $|S| = 2$ and $m = 2n-1$. Recall that $m \le 2n-1$. If $|S| = 1$, then $oc(G-S) \ge |S|$ as $oc(G-S) > 0$, which contradicts the assumption of this case. So $|S| \ge 2$. If $|S| \ge 3$, then since $n \ge 2$ we have

$$oc(G-S) = n|S| - m \ge n|S| - 2n + 1 = (n-2)(|S|-2) + |S| - 3 + |S| \ge |S|,$$

a contradiction. Hence $|S| = 2$. If $m < 2n-1$, then $oc(G-S) = n|S| - m = 2n - m \ge 2 = |S|$, and so $oc(G-S) \ge |S|$, which contradicts the assumption of this case. Therefore $|S| = 2$ and $m = 2n-1$.

Let x and y be two adjacent vertices of G. Then by (2), there exists a vertex subset T_{xy} of G such that $T_{xy} \supseteq \{x, y\}$ and $m \le n|T| - oc(G - T_{xy}) \le 2n-1 = m$. Therefore T_{xy} is a tight set of G, and so by the above observation, $T_{xy} = \{x, y\}$ and $oc(G - \{x, y\}) = 1$. In particular, $\{x, y\}$ is a tight set, and if $G - \{x, y\}$ is connected, then $G - \{x, y\}$ is an odd-cactus.

For an edge $a = uv$ of G, let $f(a)$ denote the number of vertices of a largest component of $G - \{u, v\}$. An edge e is called *extremal* if $f(e) = \max\{f(a) : a \in E(G)\}$. Let $e = xy$ be an extremal edge of G. We want to prove that $G_1 = G - \{x, y\}$ is connected. Suppose the contrary. Let B be a largest component of G_1. Let R denote the set of edges of G not incident to B and distinct from $e = xy$. Since G_1 is disconnected, the set R is not empty. Let $a = uv \in R$. Clearly if there is a vertex $z \in \{x, y\} \setminus \{u, v\}$ that is adjacent to B, then $f(a) > f(e)$, a contradiction. Since a is an arbitrary edge in R, every edge in R is incident to the same end of e, say x, and the other end y is not adjacent to B. Therefore y is an isolated vertex in $G - \{x\}$. Since an isolated vertex is an odd cactus by definition, $oc(G - \{x\}) \ge 1 > 0$. Since $n|\{x\}| - m = -n + 1 \le 0$, we have $oc(G - \{x\}) > n|\{x\}| - m$, which contradicts the definition of m.

Suppose that G_1 has a vertex v which is adjacent to exactly one of x, y. Then $G_1 - v$ has a strong $\mathcal{S}_n$–factor, say F, by (iii) of Lemma 2, and $\{x, y, v\}$ induces a 2–star S in G. Thus $F \cup S$ is a strong $\mathcal{S}_n$–factor of G. Hence we may assume that $N_G(x) \cap V(G_1) = N_G(y) \cap V(G_1) := N$. If $|N| = 1$ or N is the vertex set of a block of G_1 then G is an odd–cactus, a contradiction. Therefore there is a matching xx', yy' in G with $\{x', y'\} \subset V(G_1)$. Suppose that x' and y' are not contained in the same block of G_1. Then every component of $G_2 := G_1 - \{x', y'\}$ is a cactus with at least one and at most two even blocks. By (ii) of Lemma 2, G_2 has a strong $\mathcal{S}_n$–factor, say F, and so F together with the matching xx', yy' forms a strong $\mathcal{S}_n$–factor of G. Thus we may assume that $N \subseteq V(Q) \setminus v$ for some block Q of G_1 and $v \in V(Q)$. Then $\{x, x', v\}$ induces a 2–star S in G. Obviously every component of $G_2 := G_1 - \{v, x', y'\}$ is a cactus with exactly one even block. By (i) of Lemma 2, G_2 has a strong $\mathcal{S}_n$–factor, say F, and so F together with the star S and the edge yy' forms a strong $\mathcal{S}_n$–factor of G. □

Acknowledgment The authors would like to thank the referees for their helpful suggestions and P. Seymour for a very careful reading of the paper and very useful remarks and suggestions.

References

[1] F. Berman, D. Johnson, T. Leighton, P.W. Shor and L. Snyder, *Generalized Planar Matching*, J. of Algorithms **11** (1990) 153–184.

[2] J.A. Bondy and U.S.R. Murty, *Graph theory with applications*, North Holland, Amsterdam, 1976.

[3] G. Cornuéjols and D. Hartvigsen, An extension of matching theory, *J. Combin. Theory* **B 40** (1986) 285–296.

[4] J. Folkman and D. R. Fulkerson, *Flows in infinite graphs*, J. Combin. Theory **8** (1970) 30–44.

[5] P. Hell and D. Kirkpatrick, Packing by complete bipartite graphs, *SIAM J. Algebraic Discrete Math.* **7** (1986), 199–209.

[6] J. Akiyama and M. Kano, Factors and Factorizations of Graphs – A survey, *J. Graph Theory* **9** (1985) 1-43.

[7] A. Kelmans, Optimal packing of induced stars in a graph, *RUTCOR Research Report 26-94*, Rutgers University (1994) 1–25.

[8] M. Las Vergnas, *An extension of Tutte's 1–factor theorem*, Discrete Math. **23** (1978), 241 - 255.

[9] M. Loebl and S. Poljak, Subgraph packing – A survey, In *Topics in Combinatorics and Graph Theory: Essays in Honour of Gerhard Ringel*, Phisica Verlag, Wursburg, 1990, 491–503.

[10] A. Saito and M. Watanabe, *Partitioning graphs into induced stars*, Ars Combinatoria, **36** (1995), 3–6.