Graphs with the Balas-Uhry property

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Abstract

We characterize graphs H with the following property: Let G be a graph and F be a subgraph of G such that (i) each component of F is isomorphic to H or K_2, (ii) the order of F is maximum. and (iii) the number of H-components in F is minimum subject to (ii). Then a maximum matching of F is also a maximum matching of G. This result is motivated by an analogous property of fractional matching discovered independently by J. P. Uhry and E. Balas.

This article is motivated by a property of fractional matchings discovered by J.P. Uhry[8] and E. Balas[2]. We characterize the class of graphs to which the property can be naturally extended.

We consider finite graphs without loops or multiple edges. Let K_2 denote the complete graph of order two and C_n denote the cycle of order n. Let $\{A_i|i \in I\}$ be a set of graphs. A subgraph F of a graph G is called an $\{A\imath|i \in I\}$-subgraph if each component of F is isomorphic to one of $\{A_i|i \in I\}$. A component of F that is isomorphic to A_i is called an A_i-component. A spanning $\{A_i|i \in I\}$-subgraph is called an $\{A_i|i \in I\}$-factor. An $\{A_i|i \in I\}$-subgraph F is said to be maximum if there is no $\{A_i|i \in I\}$-subgraph F' such that $|V(F')| > |V(F)|$. A fractional matching is a $\{K_2, CC_{2n+1}|n \geq 1\}$-subgraph, and J.P. Uhry and E. Balas proved the following theorem:

Theorem 1 *([8,2]). Let F be a maximum fractional matching of a graph G having minimum number of odd cycles. Then a maximum matching of F if also a maximum matching of G.*

We say that a set $\{H_i | i \in I\}$ of graphs has the Balas-Uhry property if for every graph G, a maximum matching of s maximum $\{K_2, H_i | i \in I\}$-subgraph of G with minimum number of H-components is also a maximum matching of G, where an H-component means an H_i-component for any i. If the set $\{H_i | i \in I\}$ consists of exactly one graph H, then we say that the graph H has the Balas-Uhry property. Note that a maximum matching of a $\{K_2, H_i | i \in I\}$-subgraph F is obtained from F by replacing each H-component by a maximum matching in it.

We expect the reader to be familiar with the basic notation of matching theory (see, e.g.,[7]). A graph G is said to be perfectly matchable if G has a perfect matching. A matching with one exposed(unsaturated) vertex is called a near-perfect matching. We recall that the deficiency def(G) of G is the number of vertices unsaturated by a maximum matching. We denote by $D(G) \cup A(G) \cup C(G)$ the Gallai-Edmonds decomposition of G. Namely, every maximum matching of G contains a near-perfect matching of each component of $D(G)$, a perfect matching of each component of $C(G)$ and matches all vertices of $A(G)$ with vertices in distinct components of $D(G)$. For two sets X and Y, we write $X \triangle Y = (X/Y) \cup (Y/X)$. We prove the following two theorems:

Theorem 2 *A set $\{H_i \; i \in I\}$ of connected graphs has the Balas-Uhry peoperty if one of the following holds for every i:*
(i) H_i is perfectly matchable; or
(ii) def(H_i) $= 1$ and $C(H_i) = \emptyset$.

Theorem 3 *Let H be a connected graph having no perfect matching. If def(H) > 1 or $C(H) \neq \emptyset$, then H does not have the Balas-Uhry property.*

Theorem 2 is trivial if every H_i is a perfect matchable graph, because every maximum $\{K_2, H_i | i \in I\}$-subgraph F with minimum number of H components is already a matching. The Balas-Uhry Property has been previously proved for hypomatchable graphs (i.e., graphs H with def(H) $= 1$ and $A(H) = C(H) = \emptyset$) in [4]. Theorem 2 has been proved in a slightly different setting in [6](this proof will not appear in the final version of [6]). We will present a more detailed proof in order to make this note self-contained.

Proof of Theorem 2. We may assume that every H_i satisfies def(H_i) $=$ 1 and $C(H_i) = \emptyset$. Let G be a graph and F be a maximum $\{K_2, H_i | i \in I\}$-subgraph of G with minimum number of H-components. Let M be a maximum matching of F, which is obtained from F by replacing each H-component by a maximum matching in it. For contradiction, assume that M is not a maximum matching of G. Then there exists an alternating path

P with respect to M that joins two unsaturated vertices u and v. For any pair a and b of vertices of P, Let us denote by $P(a,b)$ the segment of P with end-vertices a and b. Now we will define a sequence

$$u = y_0, x_1, y_1, \ldots, x_n, y_n, x_{n+1} = v$$

as follows: Set $y_0 = u$, and if $y_0, x_1, y_1, \ldots, x_i, y_i$ are already defined, set x_{i+1} to be the first vertex in $P(y_i, v)$(in the direction from y_i to v) that lies in an H component distinct from that containing y_i. It there is no such vertex, set $x_{i+1} = v$ and $n = i$. Let N_i be the H-component containing x_i for every i, $1 \leq i \leq n$. Let N_0 denote the H-component containing $y_0 = u$ and N_{n+1} the H-component containing $x_{n+1} = v$, if such components exists. If $y_0, x_1, y_1, \ldots, x_i$ are defined, then set y_i to be the last vertex in $P(x_i, v)$, which is in N_i and such that there is no vertex $z \in P(x_i, y_i)$, which lies in an H-component distinct from N_i; however, if it happens that $y_i = v$, then we redefine $x_i = v$ and $n = i - 1$. Note that some N_i and N_j may coincide, but they must be distinct if $|i - j| = 1$. Furthermore, let $R_i = N_i \cup P(x_i, y_i)$ for every i, $1 \leq i \leq n$.

Claim. At least one of R_i/x_i and R_i/y_i is perfectly matchable for every i, $1 \leq i \leq n$.

We prove the claim. Assume one of x_i and y_i, say x_i, lies in $D(N_i)$. Let M_i be a perfect matching of N_i/x_i. Then a required matching of R_i/x_i is obtained by adding to M_i the K_2-components of F that are in $P(x_i, y_i)$. Assume that both $x_i, y_i \in A(N_i)$. Let us denote by a and b the vertices of N_i matched by M with x_i and y_i, respectively. Let z be the vertex of N_i exposed by M. By considering M and a maximum matching of N_i with one exposed vertex a, we observe that there exists an alternating path Q in N_i with respect to M that joins z to a. Let c be the first vertex on Q, in the direction from z to a, which lies in $P(a,b)$(c may be coincide with a or b). Then one of $Q(z,c) \cup P(c,a) \cup ax_i$ and $Q(z,a) \cup P(c,b) \cup by_i$ is an alternating path in R_i with respect to M. Let us denote this path by P'. Then $(M \cap E(R_i)) \triangle P'$ is a near perfect matching of R_i, exposing one of x_i and y_i. Therefore the claim is proved.

We distinguish two cases.

Case 1. R_1/x_1 or R_n/y_n has a perfect matching.

We may assume that R_1/x_1 has a perfect matching M_1. If u is exposed by F, then $((F/R_1) \triangle P(u, x = 1)) \cup M_1$ is a $\{K_2, H_i | i \in I\}$-subgraph of order greater than F, since its vertex set is $V(F) \cup u$. If u is saturated by F, then it lies in the H-component, which we denote by N_0. Let M_0 be the perfect matching of N_0/u used in building up M. Then

$$[(F/(N_0 \cup R_1)) \cup M_0 \cup M_1] \triangle P(u, x_1)$$

is maximum $\{K_2, H_i | i \in I\}$-subgraph with strictly fewer H-components that F.

Case 2. Neither R_1/x_1 nor R_n/y_n has a perfect matching.

By the claim, there must be some i, $1 \le i < n$, for which both R_i/y_i and R_{i+1}/x_{i+1} are perfectly matchable. Let M_i and M_{i+1} be the corresponding matchings. Then

$$[(F/(R_i \cup R_{i+1})) \triangle P(y_i, x_{i+1})] \cup M_i \cup M_{i+1}$$

is a maximum $\{k_2, H\}$-subgraph with strictly fewer H-components than F.

Consequently, we derive a contradiction in each case, and thus the theorem is proved. □

Proof of Theorem 3. Let H be a connected graph such that def$((H) \ge 2$ or def$(H) = 1$ and $C(H) \ne \emptyset$. We shall show that there exists a graph G such that a maximum matching of a maximum $\{K_2, H\}$-subgraph of G with minimum number of H-components is not a maximum matching of G. We distinguish three cases.

Case 1. def$(H) \ge 3$.

Let u be a vertex in $D(H)$, and let G be a graph obtained from H by adding a new vertex w and by joining u to w by a new edge. Then def$(G) =$ def$(H) - 1 \ge 2$, and H is a maximum $\{K_2, H\}$-subgraph of G with minimum number of H-components. It is clear that a maximum matching of H is not a maximum matching of G.

Case 2. def$(H) = 2$.

Let H_1 and H_2 be two copies of H, and let $u_1 \in D(H)$ and $w_1 \in D(H_2)$. We construct a graph G from H_1 and H_2 by joining u_1 to w_1 by a new edge. It is easy to see that def$(G) = 2$ and G has a $\{K_2, H\}$-factor $H_1 \cup H_2$. Suppose that G has a $\{K_2, H\}$-factor F that contains exactly one H-component R. Let M be a maximum matching of R. If M does not contain the edge u_1w_1, then the defect of M(i.e., the number of vertices of R unsaturated by M) is greater than or equal to four, and thus a maximum matching of F is not a maximum matching of G. Hence we may assume that M contains the edge u_1w_1. Then $N_i = (M \cap H_i) \cup (K_2 -$ components of F in $H_i)$, $i = 1, 2$ is a maximum matching of H_i. Let $u_1, u_2 \in V(H_1)$ and $w_1, w_2 \in V(H_2)$ be the unsaturated vertices of N_1 and N_2, respectively. Let l be the minimum number of bridges(cut-edges) in a path that joins the two unsaturated vertices of a maximum matching of H. Let P be a path in R that joins the two unsaturated vertices u_2 and w_2 of M. Since a path $P \cap H_1$ joins the two unsaturated vertices u_1 and u_2 of a maximum matching N_1 of H_1, $P \cap H_1$ contains at least l bridges. Similarly, a path $P \cap H_2$ contains at least l bridges. Hence the path P contains at least $2l+1$ bridges. Therefore $l \geq 2l+1$ since M is an arbitrary maximum matching of R. This is a contradiction. Consequently, we may assume that $H_1 \cup H_2$ is a unique $\{K_2, H\}$-factor of G, and thus G possesses the required property.

Case 3. def(H) $= 1$ and $C(H) \neq \emptyset$.

Let $H_1,\ H_2,\ H_3$ be three copies of H. Let u and w be two vertices in $C(H_2)$ such that the edge uw is contained in some maximum matching of H_2, and the distance between u and $D(H_2)$ is a maximum among all vertices in $C(H_2)$(i.e., $d(u, D(H_2)) \geq d(x, D(H_2))$ for all $x \in C(H_2)$). Let a and b be vertices in $D(H_2)$ and $D(H_3)$, respectively. We construct a graph G from $H_1,\ H_2,\ H_3$ by joining u to a and w to b by new edges. We shall show that G has a unique $\{K_2, H\}$-factor $H_1 \cup H_2 \cup H_3$.

It is easy to see that def(H) $= 1$. Suppose that G has a $\{K_2, H\}$-factor F, which is not $H_1 \cup H_2 \cup H_3$. We first assume that F contains exactly one H-component R. Since $V(G)/V(R)$ are covered by K_2-components of F, R contains a, b, u, w. Let N be a maximum matching of R. If N does not contain the edge ua, then def(R) $\geq$ def(H_1) + def(H_2) $>$ def(h), a contradiction. Thus N contains ua, and similarly, N contains wb. Since N covers all the vertices in $(V(H_1) \cap R) \cup (V(H_3) \cap R) \cup \{u, w\}$, we have that $D(R) \subset V(H_2)$ and so $D(R) = D(H_2)$. It follows that $a \in C(R)$ and the distance between a and $D(R)$ is greater than that between u and $D(H_2)$, which contradicts the chioce of u.

We next suppose that F contains two H-components R_1 and R_2. Since $V(G)/(V(R_1) \cup V(R_2))$ are covered by K_2-components and $|V(G)| \equiv |V(H) \equiv 1(\text{mod} = 2)$, we have a contradiction. Therefore F contains three H-components, and so $F = H_1 \cup H_2 \cup H_3$.

Since def($H_1 \cup H_2 \cup H_3$) $= 3 >$ def(G), G possesses the required property. □

It is well known that a maximum fractional matching, and also a maximum $\{K_2, H\}$-subgraph for some graphs H with $C(H) = \emptyset$ and def(H) $=$ 1(see[3,4]), can be found in polynomial time. From that point of view, it might be surprising that the polynomial time algorithm cannot be extended to all graphs with the Balas-uhry property, but only to its proper subclass. The class of connected graphs H for which the $\{K_2, H\}$-factor problem is polynomially solvable has been described in [5, 6].

Let us also remark that a survey of some other results on component factors can be found in [1].

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