Spanning trees fixed by automorphisms of a graph

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Abstract

Let G be a finite graph and A be a subgraph of $Aut(G)$. We give a necessary and sufficient condition for the graph G to have an A-invariant spanning tree.

1 Introduction

We consider a finite graph G which has neither loops nor multiple edges. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G, respectively. An edge joining two vertices v and w is denoted by vw or wv. An automorphisms α of G is permutation on $V(G)$ that preserves adjacency, that is, if $e = vw$ is an edge of G then $\alpha(e) = \alpha(v)\alpha(w)$ is also an edge of G. The set of automorphisms of G forms the automorphism graph $Aut(G)$. Let $\alpha \in Aut(G)$ and Abe a subgraph of $Aut(G)$. For a subset X of $E(G)$, we write $\alpha(X) = \{\alpha(e)|e \in X\} \subseteq E(G)$, and say that X is A-invariant if $\sigma(X) = X$ for all $\sigma \in A$. We write $F(A) = \{v \in V(G)|\alpha(v) = v \text{ for all } \alpha \in A\}$. We denote by $G[F(A)]$ the subgraph of G induced by $F(A)$. For a vertex x of G, define the subgraph A_x of A by $A_x = \{\alpha \in A|\alpha(x) = x\}$ Other notation and definitions not defined in this paper can be found in [1].

We consider the following problem. Let $\mathfrak{F}$ be a property of graphs and A be a subgroup of $Aut(G)$. When does the graph G have an A-invariant subgraphs that possesses the property $\mathfrak{F}$? In this paper we prove the following theorem.

Theorem 1 *Let G be a connected graph, and A be a subgroup of $Aut(G)$. Then G has an A-invariant spanning tree if and only if one of the following conditions is satisfied:*

(i) $F(A) \neq \emptyset$ and the induced subgraph $G[F(A_x)]$ is connected for every vertex x of G; or

(ii) $F(A) = \emptyset$ and A fixes an edge uw such that $F(A_x) \supset \{u, w\}$ for all $x \in V(G)$. Moreover, for every verte x of G, the induced subgraph $G[F(A_x)]$ is connected.

Note that is A satisfies (ii) in the above theorem, then $|V(G)|$ is even, since every orbit of A acting on $V(G)$ has an even length $|A : A_x| = |A : A_u||A_u : A_x| = 2|A_u : A_x|$.

We now give examples. Let A be a subgroup of $Aut(K_m)$, where K_m denotes the complete graph on m vertices. Then it is easily shown by the theorem that K_{2n+1} contains an A-invariant spanning tree if and only if A fixes at least one vertex. Furthermore, it also follows immediately from the theorem that K_{2n} contains an A-invariant spanning tree if and only if (a) A fixes at least one vertex, or (b) A fixes an edge uw such that $F(A_x) \supset \{u, w\}$ for all vertices x of K_{2n}(i.e. in the cycle decomposition of every $\alpha \in A$ acting on $V(G)$ with $\alpha(u) = w$, each cycle has an even length).

2 Proof of Theorem

Proof of necessity. Let T be an A-invariant spanning tree of G, where we regard T sa a subset of $E(G)$. Let B be any subgraph of A with $F(B) \neq \emptyset$. Let x and y be any two distinct vertices of $F(B)$, and let $P(x, y)$ denote the unique path in T from x to y. Then B fixes $P(x, y)$, and so B fixes all the vertices in $P(x, y)$. Hence $G[F(B)]$ is connected. In particular, if $F(A) \neq \emptyset$, then condition (i) holds.

We now assume $F(A) = \emptyset$. Since the center of T, which is clearly A-invariant, is K_1 or K_2, we obtain that the center of T is K_2 and that A fixes the edge uw contained in the center of T. Let B be a subgroup of A. Suppose u is not contained in $F(B)$. Take $\beta \in B$ such that $\beta(u) = w$ and $\beta(w) = u$. Then β cannot fix any other vertex since $d(x, u) \neq d(x, w)$ for any other vertex x in the tree T, where $d(x, y)$ denotes the distance between x and y. Therefore if $F(B) \neq \emptyset$, then $F(B)$ contains both u and w. Consequently, if $F(A) = \emptyset$ then condition (ii) follows.

Proof of the sufficiency of (i). For a subgroup B of A, we denote the order of B by $|B|$.

We assume that condition (i) holds. Put $X_0 = F(A)$, and let $X_1, X_2, \ldots, X_m$ be the orbits of the permutation group A acting on $V(G)/X_0$. We construct a digraph D with vertex set $V(D) = \{X_0, X_1, \ldots, X_m\}$ from G an follows. For any two vertices X and Y of D, (X, Y) is an arc of D if and only if there exists an edge xy of G such that $x \in X$, $y \in Y$ and $A_x \supset A_y$(i.e. A_x fixes x). We shall show that D is connected and has a rooted spanning tree T^* possessing the property that for every vertex X, $X \neq X_0$, there exists a directed path from X_0 to X in T^*. In order to prove the connectivity of D and the existence of T^*, it suffices to show that the following two statements holds.

(1) For each vertex $X \neq X_0$ of D, the in-degree of X is positive

(2) For each strongly connected component C of D with $X_0 \notin V(C)$, there exist vertices $X \in V(C)$ and $X' \notin V(C)$ such that (X', X) is an arc of D.

We first prove (1). Let $x \in X$. Since the induced subgraph $G[F(A_x)]$ is connected, there exists a path $x, a, \ldots, d, p, \ldots, r, s$ in $G[F(A_x)]$ such that $xa, \ldots, dp, \ldots, rs$ are edges, $\{x, a, \ldots, d\} \subset X$, $p \in X' \neq X$ and $s \in X_0$. Since A_x is included in $A_a, \ldots, A_s$ and $|A_x| = |A_a| = \cdots = |A_d|$(as X is an orbit), we obtain $A_x = A_d$. Hence (X', X) is an arc of D as $pd \in E(G)$ and $A_d = A_x \subset A_p$. Therefore (1) is proved.

I is immediately that if (X_i, X_j) is an arc of D, then for any vertices $a \in X_i$ and $b \in X_j$, we have $|A_a| \geq |A_b|$. Thus, if X_i and X_j are vertices of a strongly connected component of D, then for all vertices $a \in X_i$ and $b \in X_j$, we have $|A_a| = |A_b|$. We now prove (2). Take a vertex $x \in X$, $X \in V(C)$, and put $V(C) = \{X_1, X_2, \ldots, X_n\}$. Then there exists a path $x, a, \ldots, c, p, \ldots, r, s$ in $G[F(A_n)]$ such that $xa, \ldots, rs$ are edges of G, $\{x, a, \ldots, c\} \subset X_1 \cup \cdots \cup X_n$, $c \in X_k$, $X_k \in V(C)$, $p \in X'$, $X' \notin V(C)$ and $s \in X_0$. Thus (X', X_k) is an arc of D as $A_c = A_x \subset A_p$. Hence (2) holds.

Let T^* be a rooted spanning tree of D given above. For each arc (X, Y) of T^*, we take exactly one edge $e = xy$ of G such that $x \in X$, $y \in Y$ and $A_x \supset A_y$, and let $\{e_1, \ldots, e_m\}$ be the set of such edges. Set

$$T = (\text{a spanning tree of } \ G[X_0]) \cup \{\alpha((e_i)|1 \leq i \leq m, \alpha \in A\}.$$

Then it is obvious that T is an A-invariant connected spanning subgraph of G. Suppose T contains a cycle. Then there exists an edge $e = xy \in \{e_1, \ldots, e_m\}$ such that $\sigma(x) \neq \tau(x)$ and $\sigma(y) = \tau(y)$ for some $\sigma, \tau \in A$. Then $\tau^{-1}\sigma(x) \neq x$ and $\tau^{-1}\sigma(y) = y$, which contradicts $A_x \supset A_y$. Therefore, T contains no cycle, and we can conciude that T is a desired A-invariant spanning tree of G.

Proof of sufficiency of (ii). We construct a new graph G' from G be inserting a new vertex v of degree 2 into the edge uw. Then $V(G') = V(G) \cup$

$\{v\}$ and $E(G') = (E(G)/\{uw\}) \cup \{uv, vw\}$. For every $\alpha \in A$, define the permutation α' acting on $V(G')$ by $\alpha'(x) = \alpha(x)$ for all $x \in V(G)$ and $\alpha(v) = v$. Then α' is an automorphism of G'. condition (ii) guarantees that G' and $A' = \{\alpha' \in Aut(G') | \alpha \in A\}$ satisfy condition (i). Hence G' has an A'-invariant spanning tree T'. It is clear that T' contains edges uv and uw. Consequently, G has an A-invariant spanning tree $T = (T'/\{uv, iw\}) \cup \{uw\}$. $\square$

References

[1] M. Behzad, G. CHartrand, and L. Lesniak-Foster, Graphs and Digraphs, *(Prindle, Weber and Schmidt, Boston,1979*