Star factors with given properties

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Abstract

A spanning subgraph F of a graph G is called a $\{K_{1,1}, \ldots, K_{1,n}\}$-factor if each component of F is isomorphic to one of $\{K_{1,1}, \ldots, K_{1,n}\}$, where $K_{1,k}$ denotes the star of order $k+1$. Among other results, we show that for a graph G, the following two statement are equivalent:
(1) every edge e of G possesses the property that some $\{K_{1,1}, \ldots, K_{1,n}\}$-factor of G has a component which is isomorphic to $K_{1,1}$ or $K_{1,2}$ and contain e;
(2) for all $S \subset V(G)$, the number of isolated vertices of $G - S$ is at most $n|S| - \varepsilon(S)$, where $n \geq 2$ and $\varepsilon(S)$ is defined to be $2n-1$ when the subgraph induced by S contains an edge, and to be 0 otherwise.

1 Preliminaries.

We deal only with finite graphs which have neither multiple edges nor loops. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$, respectively. We denote by $I(G)$ the set of isolated vertices of G, and write $i(G) = |I(G)|$. Let $o(G)$ denote the number of odd components of G. For a subset S of $V(G)$, we denote by $G[S]$ the subgraph of G induced by S and by $G - S$ the subgraph of G obtained from G by deleting the vertices in S together with their incident edges. Note that $G - S = G[V(G)/S]$. The neighborhood $\Gamma_G(S)$ is defined to be the set of vertices of G which are adjacent to at least one vertex in S. If S consists of a single vertex v of G, we write $G - v$ for $G - \{v\}$, and $\Gamma_G(v)$ for $\Gamma_G(\{v\})$. The degree of a vertex v of G, which is equal to $|\Gamma_G(v)|$, is denoted by $\deg_G(v)$.

Let $K_{1,k}$ denote the star of order $k+1$, that is, the complete bipartite graph with partite sets of order 1 and k. A spanning subgraph F of G is called a $\{K_{1,1}, \ldots, K_{1,n}\}$-factor if each component of F is isomorphic to one of $\{K_{1,1}, \ldots, K_{1,n}\}$. For a positive integer n, a $[1,n]$-factor is a spanning subgraph F such that $1 \le \deg_F(v) \le n$ for all vertices v of F.

There exists numerous results on degree factors containing or not containing given edges. For example, Little [6](or see [7]) proved that for every edge e of a graph G, G has a 1-factor which contains e, if and only if $o(G-S) \le |S|$ for all $S \subset V(G)$ and $o(G-S) = |S|$ implies that the induced subgraph $G[S]$ has no edges. Berge and Las Vergnas [3] proved that for every edge e of a graph G, G has a $[1,n]$-factor($n \ge 2$) which contains e, if and only if $i(G-S) \le n|S| - \varepsilon(S)$ for all $S \subset V(G)$, where $\varepsilon(S) = 2$ if $G[S]$ has an edge; $\varepsilon(S) = 1$ if $G[S]$ has no edges, $G - S$ has a edge and $S \ne \emptyset$; and $\varepsilon(S) = 0$ otherwise. For other results of this type, we refer the reader to [5] and the survey article [1]. However, there are few results of this type on component factors. Our main theorem gives a sufficient conditions for a graph to have a $\{K_{1,1}, \ldots, K_{1,n}\}$-factor which contains an arbitrarily given edge.

2 The results.

The following theorem is our main result.

Theorem 1 *Let G be a graph and $n \ge 2$ be an integer. Then the following two statements are equivalent:*
(i) every edge e of G possesses the property that some $\{K_{1,1}, \ldots, K_{1,n}\}$-factor of G has a component which is isomorphic to $K_{1,1}$ or $K_{1,2}$ and contains e;
(ii) for all $S \subset V(G)$, we have

$$i(G - S) \le n|S| - \varepsilon_1(S), \tag{1}$$

where

$$\varepsilon_1(S) = \begin{cases} 2n-1 & \text{if } G[S] \text{ contains an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

This theorem in particular implies that the condition (1) in the Theorem is a sufficient condition for a graph G to have a $\{K_{1,1}, \ldots, K_{1,n}\}$-factor which contains an arbitrarily given edge. This sufficient condition (1) is beat possible in the sense that for every integer $n \ge 2$, there exists a graph G and an edge e of G such that G has no $\{K_{1,1}, \ldots, K_{1,n}\}$-factor containing e but satisfies the following inequality for every $S \subset V(G)$:

$$i(G - S) \leq n|S| - \varepsilon_2(S),$$

where

$$\varepsilon_2(S) = \begin{cases} 2n-2 & \text{if } G[S] \text{ contians an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

An example of such a graph is shown in Figure 1.

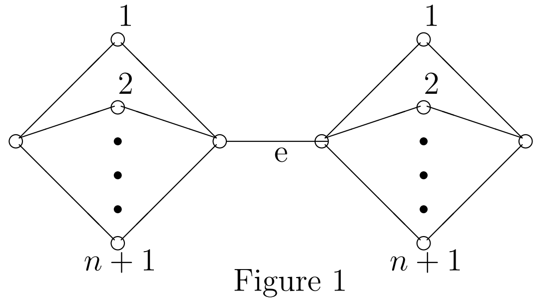

Figure 1

Our proof of the Theorem essentially depends on Lemma 1, which is due to Amahashi and Kano[2], Lemma 2 is an easy consequence of Hall's Marriahe theorem [4].

Lemma 1 *Let $n \geq 2$ be an integer. Then a graph G has a $\{K_{1,1}, \ldots, K_{1,n}\}$-factor is and only if $i(G - S) \leq n|S|$ for all $S \subset V(G)$.*

Lemma 2 *Let G be a bipartite graph with partite sets A and B, and n be a positive integer. If $|\Gamma_G(X)| \geq n|X|$ for all $\emptyset \neq X \subseteq A$, then G has a factor F such that $\deg_F(a) - n$ for all $a \in A$ and $\deg_F(b) \leq 1$ for all $b \in B$.*

Lemma 3 *Let G be a graph and $n \geq 2$ be an integer. Assume that $i(G-S) \leq n|S| + 1$ for all $S \subset V(G)$, and that there exists a subset S of $V(G)$ for which $i(G - S) = n|S| + 1$. Let S_0 be a minimal subset of $V(G)$ such that $i(G - S_0) = n|S_0| + 1$. Then for every isolated vertex v of $G - S_0$, $G - v$ has a $\{K_{1,1}, \ldots, K_{1,n}\}$-factor.*

Proof. Let v be an arbitrary isolated vertex of $G - S_0$. Suppose $S_0 \neq \emptyset$. Then v is the unique isolated vertex of G, and for every $\emptyset \neq S \subset V(G)/v$, we have $i((G-v)-S) = i(G-S)-1 \leq n|S|$. Hence $G-v$ has a $\{K_{1,1}, \ldots, K_{1,n}\}$-factor by Lemma 1. Thus, we may assume $S_0 \neq \emptyset$. In particular, G has no isolated vertices, and so $\Gamma_G(S_0) \supseteq I(G - S_0)$.

Let $H = G - (S_0 \cup I(G - S_0))$. Then H has a $\{K_{1,1}, \ldots, K_{1,n}\}$-factor since otherwise H has a subset $T \subset V(H)$ with $i(H - T) < n|T|$ by Lemma 1, and so $i(G - (S_0 \cup T)) = i(H - T) + i(G - S_0) \geq n|S_0 \cup T| + 2$, a contradiction. Therefore, in order to prove this lemma, it suffices to show that $G[S_0 \cup I(G - S_0)] - v$ has a $\{K_{1,1}, \ldots, K_{1,n}\}$-factor.

Let K be the bipartite graph with partite sets S_0 and $I(G - S_0)$, in which two vertices $x \in S_0$ and $y \in I(G - S_0)$ are joined by an edge if and only if x and y are adjacent in G. Note that K is a subgraph of $G[S_0 \cup I(G - S_0)]$. Suppose that $|\Gamma_K(X)| \leq n|X|$ for some $\emptyset \neq X \subseteq S_0$. Then $X \neq X \subset S_0$. Then $X \neq S_0$ and $i(G - (S_0/X)) \geq i(G - S_0) - |\Gamma_k(X)| \geq n|S_0| + 1 - n|X| = n|S_0/X| + 1$. Hence $i(G - (S_/X)) = n|S_0/X| + 1$ by the assumption of this lemma. This contradicts our minimal choice of S_0. Therefore $|\Gamma_K(X)| \geq n|X| + 1$ for all $\emptyset \neq X \subset S_0$. Consequently, we have $|\Gamma_{K-v}(X)| \geq n|X|$ for all $X \subset S_0$. Hence by Lemma 2, $K - v$ has a factor F such that $\deg_F(x) = n$ for all $x \in S_0$ and $\deg_F(y) = 1$ for all $y \in I(G - S_0)/v$, and thus the lemma is proved. □

Proof of the Theorem. We first prove that (i) implies (ii). Let $S \subset V(G)$. If $G[S]$ has no edges, then $i(G - S) \leq n|S|$ by Lemma 1. Hence we may assume that $G[S]$ has an edge $e = ab$ joining two vertices a and b. Then G has a $\{K_{1,1}, \ldots, K_{1,n}\}$-factor F such that F contains e and the component of F containing e is $K_{1,1}$ or $K_{1,2}$. Thus we have

$$i(G - S) \leq i(F - S) \leq n(|S| - 2) + 1 = n|S| - \varepsilon_1(S).$$

We next prove that (ii) implies (i). Let $e = ab$ be an edge of G. Let $H = G - \{a, b\}$ and S be a subset of $V(H)$. Then $i(H - S) = i(G - (S \cup \{a, b\})) \leq n(|S| + 2) - (2n - 1) = n|S| + 1$, since $G[S \cup \{a, b\}]$ contains the edge e. If $i(H - S) \leq n|S|$ for all $S \subset V(H)$, then H has a $\{K_{1,1}, \ldots, K_{1,n}\}$-factor F, and thus $F \cup \{e\}$ is a factor of G with the desired properties. Hence we may assume that $i(H - S) = n|S| + 1$ for some $S \subset V(H)$. Choose a minimal subset $S_0 \subset V(H)$ such that $i(H - S_0) = n|S_0| + 1$. If G has no edge joining a vertex of $I(G - S_0)$ to either of $\{a, b\}$, then $i(G - S_0) = i(H - S_0) = n|S_0|| + 1$, a contradiction. Therefore, there exists a vertex v in $I(H - S_0)$ which is adjacent to a or b, say a. By Lemma 3, $H - v$ has a $\{K_{1,1}, \ldots, K_{1,n}\}$-factor F, and thus we obtain a desired factor of G from F by adding the two edges e and va. □

As an immediate corollary to the Theorem, we obtain:

Corollary 1 (8) . *For a graph G, the following are equivalent:*

(i) for every edge e of a graph G, G has a $\{K_{1,1}, \ldots, K_{1,n}\}$-factor which contains e;

(ii) $i(G-S) \le 2|S| - \varepsilon_3(S)$ *for all* $S \subset V(G)$, *where*

$$\varepsilon_3(S) = \begin{cases} 3 & \textit{if } G[S] \textit{ contains an edge,} \\ 0 & \textit{otherwise.} \end{cases}$$

Also the following statement can be proved as in the first half of the Theorem 1: Let G be a graph and $n \ge 2$ be an integer. If for every edge e of G, G has a $\{K_{1,1}, \dots, K_{1,n}\}$-factor which contains e, then

$$i(G-S) \le \varepsilon_4(S) \quad \text{for all} \quad S \subset V(G),$$

where

$$\varepsilon_4(S) = \begin{cases} n+1 & \text{if } G[S] \text{ contains an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that there exists a graph G each of whose edge is contained in a $\{K_{1,1}, \dots, K_{1,n}\}$-factor and which satisfied (2) with equality for some S with $E(G[S]) \ne \emptyset$.

As for a $\{K_{1,1}, \dots, K_{1,n}\}$-factor not containing a given edge, the following proposition holds. We call an edge e of a graph an isolated edge if both ends of e have degree one and a terminal edge if exactly one of ends of e has degree one.

Proposition 1 *Let* G *be a graph and* $n \ge 2$ *be an integer. Then the following are equivalent:*

(i) for every edge e *of* G, G *has a* $\{K_{1,1}, \dots, K_{1,n}\}$*-factor which does not contain* e*;*

(ii)

$$i(G-S) \ge n|S| - \varepsilon_5(S) \quad \textit{for all} \quad S \subset V(G),$$

where

$$\varepsilon_5(S) = \begin{cases} 2 & \textit{if } G-S \textit{ contians an isolated edge,} \\ 1 & \textit{if } G-S \textit{ contains a terminal edge but no isolated edge,} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. For an edge e of G, let $G-e$ denote the graph obtained from G by deleting e. Then by Lemma 1, the statement (i) is equivalent to the condition that for each $S \subset V(G)$, we have $i((G-e)-S) \le n|S|$ for every edge e of G, and (ii) is nothing but a mere translation of this condition. □

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