Ranking the vertices of an r-partite paired comparison digraph

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Abstract

A paired comparison digraph D is a weighted digraph in which the sum of the weights of arcs, if any, joining two vertices exactly one. A one-to-one mapping from $V(D)$ onto $\{1, 2, \ldots, |V(D)|\}$ is called a ranking of D, and a ranking α of D is optimal if the backward length of α is minimum. We say that D is r-partite if $V(D)$ can be partitioned into $V_1 \cup \cdots \cup V_r$ so that every arc of D joining a vertex of V_i to a vertex of V_j, where $i \neq j$. We show that we can easily obtain all the optimal ranking of a certain r-partite paired comparison digraph.

1 Introduction

We consider a weighted digraph D with vertex set $V(D)$ and arc set $A(D)$. We denote the weight of an arc vw by $\varepsilon(vw)$, where vw joins a vertex v to a vertex w. A weighted digraph D is called a paired comparison digraph(or briefly PCD) is D satisfies the following three conditions:

(i) $0 < \varepsilon(vw) \leq 1$ for every $vw \in A(D)$.

(ii) $\varepsilon(vw) + \varepsilon(wv) = 1$ if $vw,\ wv \in A(D)$.

(iii) $\varepsilon(vw) = 1$ if $vw \in A(D)$ and $wv \notin A(D)$.

A digraph D can be considered as a PCD if we set the weight of each arc of D as follows:

(iv) $\varepsilon(vw) = \varepsilon(wv) = 0.5$ if $vw,\ wv \in A(D)$, and

(v) $\varepsilon(vw) = 1$ if $vw \in A(D)$ and $wv \notin A(D)$.

A PCD D is called an r-partite PCD if $V(D)$ can be partitioned into $V(D) = V_1 \cup \cdots \cup V_r$ so that

(vi) if $v, w \in V_i$, then v and w are not joined by an arc for all i, $1 \leq i \leq r$.

An r-partite PCD D with partition $V(D) = V_1 \cup \cdots \cup V_r$ is called an r-partite complete PCD(see Fig.1) if

(vii) any two vertices $v \in V_i$ and $w \in V_j (i \neq j)$ are joined by at least one arc.

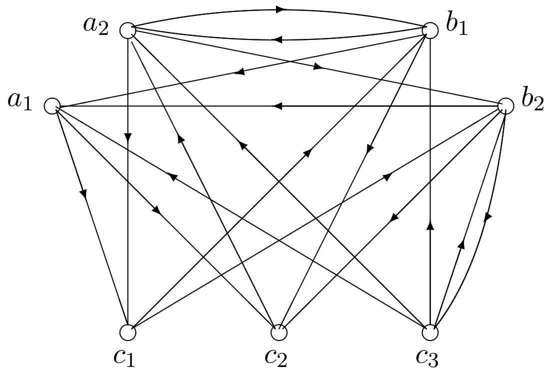

Fig.1. A 3-partite complete PCD. The weight of each arc is given by (iv) or (v)(i.e. 1 or 5).

An r-partite complete PCD can represent the outcomes of plays between r teams with all pairs of plays of different teams, in which wa allow ties ($\varepsilon(vw) = \varepsilon(wv) = 0.5$) and also more than one plays between the same players($\varepsilon(vw) = \varepsilon_1 > 0$ and $\varepsilon(wv) = 1 - \varepsilon_1 > 0$ mean that v beats w with rate ε_1 and w beats v with rate $1 - \varepsilon_1$).

We introduced a new method of ranking the vertices of a PCD in [2], and defined optimal rankings, by which we can rank the vertices of a PCD. In this paper, we shall show that the optimal rankings of an r-partite complete PCD can be easily obtained. Moreover, if the number of uncompared pairs of an r-partite PCD is small, then we can easily obtain the optimal rankings of it. Note that it is an Np-complete problem to obtain the optimal rankings of any PCD(see [2]).

We now brifly explain our method of ranking. Let D be a PCD with n vertices. A ranking α of D is a one-to-one mapping from $V(D)$ onto $\{1, 2, \ldots, n\}$. For a ranking α of D, the image $\alpha(v)$ of v is called the ranking of v defined by α. An arc wv such that $\alpha(v) < \alpha(w)$ is called a backward arc of α, and we write $B(\alpha)$ for the set of all backward arcs of α, that is,

$$B(\alpha) = \{wv \in A(D) | \alpha(v) < \alpha(w)\}.$$

We define the backward length $||B(\alpha)||$ of α by

$$||B(\alpha)|| = \sum_{wv \in B(\alpha)} \varepsilon(wv)(\alpha(w) - \alpha(v)).$$

A ranking α of D is said to be optimal if the backward length of α is minimum among the backward lengths of all rankings of D. We denote by

$OR(D)$ the set of all optimal rankings of D, and our method of ranking the vertices of D is one making use of

$$\pi(v) = \frac{1}{|OR(D)|} \sum_{\alpha \in OR(D)} \alpha(v) \quad \text{for all} \quad v \in V(D).$$

Of course, v is stronger than w if $\pi(v) < \pi(w)$. In particular, the champion is the player whose value of π ia minimum.

We denote a ranking α of D by $\alpha = [v_1, v_2, \ldots, v_n]$ if $V(D) = \{v_1, \ldots, v_n\}$ and $\alpha(v_i) = i$ for all i, $1 \leq i \leq n$. For a ranking α of D and a subset X of $V(D)$, we define the restriction $\alpha|_X : X \mapsto \{1, 2, \ldots, |X|\}$ by

$$\alpha|_X(x) = \sharp\{v \in X | \alpha(v) \leq \alpha(x)\} \quad \text{for all} \quad x \in X.$$

The score(positive score)$\sigma^+(v)$ of $v \in V(D)$ is the sum of the weights of all arcs vw, $w \in V(D)/\{v\}$. Then

$$\sigma^+(v) = \sigma^+_D(v) = \sum_{vw \in A(D)} \varepsilon(vw).$$

The negative score $\sigma^-(v)$ can be defined analogously. For a ranking α of an r-partite PCD D with partition $V(D) = V_1 \cup \cdots \cup V_r$, we define a function $\Psi(\alpha, v)$ on $V(D)$ by

$$\Psi(\alpha, v) = \Psi_D(\alpha, v) = \sigma^+(v) + |V_t| - \alpha|_{V_t}(v),$$

where $v \in V_t$ for all $v \in V(D)$. Our main theorem is the following:

Theorem 1 *Let D be an r-partite complete PCD with partite sets $V_1, \ldots, V_r$, and put $|V(D)| = n$. Then a ranking $\alpha = [v_1, \ldots, v_n]$ of D is optimal if and only if the following two conditions hold.*

(1) For every $U \in \{V_1, \ldots, V_n\}$, put $\alpha|_U = [u_1, \ldots, u_k]$. Then $\sigma^+(u_1) \geq \cdots \geq \sigma^+(u_k)$.

(2) $\Psi(\alpha, v_1) \geq \cdots \geq \Psi(\alpha, v_n)$.

For example, let D be a 3-partite complete PCD given in Fig.1, and α be an optimal ranking of D. Set $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$ and $C = \{c_1, c_2, C_3\}$. Since $\sigma^+(a_1) = 2$C $\sigma^+(a_2) = 2.5$, $\sigma^+(b_1) = \sigma^+(b_2) = 2.5$, $\sigma^+(c_1) = 3.5$C $\sigma^+(c_2) = 2$, and $\sigma^+(c_3) = 2$, we have $\alpha|_A = [a_2, a_1]$, $\alpha|_B = [b_1, b_2]$ of $[b_2, b_1]$, and $\alpha|_C = [c_1, c_3, c_2]$. Thus $\Psi(\alpha, a_2) = 3.5$, $\Psi(\alpha, a_1) = 2$, $\Psi(\alpha, b_1) = 3.5$, $\Psi(\alpha, b_2) = 2.5$(or $\Psi(\alpha, b_2) = 3.5$, and $\Psi(\alpha, b_1) = 2.5$), $\Psi(\alpha, c_1) = 5.5$, $\Psi(\alpha, c_3) = 3$ and $\Psi(\alpha, c_2) = 1$. Therefore $\alpha = [c_1, u, w, c_3, b_2, a_1, c_2]$, where

$\{u, w\} = \{a_2, b_1\}$, or $\alpha = [c_1, u, w, c_3, b_1, a_1, c_2]$, where $\{u, w\} = \{a_2, b_2\}$. In particular, $|OR(D)| = 4$.

We conclude this section by giving a conjecture. A PCD D is said to be ranking equal if $\pi(v) = (|V(D)| + 1)/2$ for all $v \in V(D)$. A PCD D is said to be balanced(regular) if $\sigma^+(v) = \sigma^-(v)$ for all $v \in V(D)$.

Conjecture. Let D be a PCD with the weight of every arc 1(i.e. D is an oriented digraph.) Then D is ranking equal if and only if D is balanced.

We can prove that every balanced PCD is ranking equal, and show that the condition that the weight of every arc is 1 is necessary(see [2]).

2 Proof of Theorem 1

Let D be a PCD. We define a function $\mu : V(D) \times V(D) \mapsto \{0, 1\}$ by

$$\mu(vw) = \mu(wv) = \begin{cases} 1 & \text{if } v \text{ and } w \text{ are joined by an arc,} \\ 0 & \text{otherwise.} \end{cases}$$

For a ranking α of D such that $\alpha(v) = k$ and $\alpha(w) = k + m > k$, let α^k_m denote the ranking defined by

$$\alpha^k_m(x) = \begin{cases} k + m & \text{if } x = v, \\ k & \text{if } x = w, \\ \alpha(x) & \text{otherwise.} \end{cases}$$

Lemma 1[2]. Let α be a ranking of D such that $\alpha(v) = k$ and $\alpha(w) = k + m > k$. Then

$$\begin{aligned} ||B(\alpha^k_m)|| - ||B(\alpha)|| &= m(\sigma^+(v) - \sigma^+(w)) \\ &+ m(\sum_{k+m<\alpha(x)} (\mu(wx) - \mu(vx)) + \sum_{k<\alpha(y)<k+m} (\alpha(y) - k)(\mu(wy) - \mu(vy))) \end{aligned}$$

Lemma 2. Let D be an r-partite complete PCD with partite sets $V_1, \ldots, V_r$, and let X, $Y \in \{V_1, \ldots, V_r\}$, $X \neq Y$. Then

(1) If $v, w \in X$, $\alpha(w) = k + m > k$, then

$$||B(\alpha^k_m) - ||B(\alpha)|| = m(\sigma^+(v) - \sigma^+(w)).$$

(2) If $v \in X$, $w \in Y$, $\alpha(v) = k$ and $\alpha(w) = k + m > k$ and there is no vertex $u \in X \cup Y$ such that $\alpha(v) < \alpha(u)\alpha(w)$, then

$$||B(\alpha^k_m) - ||B(\alpha)|| = m(\Psi(\alpha, v) - \Psi(\alpha, w)).$$

Proof. (1) It is obvious that $\mu(vz) = \mu(wz)$ for all $z \in V(D)$. Hence (1) is an easy consequence of Lemma 1.

(2) Since $\mu(vz) = \mu(wz)$ for all $z \in V(D)/(X \cup Y)$, $\mu(vx) = 0$ and $\mu(wx) = 1$ for all $x \in X$, and $\mu(vy) = 1$ and $\mu(wy) = 0$ for all $y \in Y$, we have by Lemma 1 that

$$\begin{aligned} ||B(\alpha^k_m) - ||B(\alpha)|| &= m(\sigma^+(v) - \sigma^+(w)) + m(\sharp\{x \in X | k+m < \alpha(x)\} \\ &\qquad - \sharp\{y \in Y | k+m < \alpha(y)\}) \\ &= m(\sigma^+(v) + |X| - \alpha|_X(v) - (\sigma^+(w) + |Y| - \alpha|_Y(w))) \\ &= m(\Psi(\alpha, v) - \Psi(\alpha, w)). \quad \square \end{aligned}$$

Proof of Theorem 1. We first prove the necessity. Assume α is an optimal ranking of D. Then (1) of the theorem fpllpws immediately from Lemma 2. We next prove (2). Suppose that there exist $v, w \in V(D)$ such that $\alpha(v) < \alpha(w)$ and $\Psi(\alpha, v) < \Psi(\alpha, w)$. By (1), we may assume $v \in V_s$, $w \in V_t$ and $s \neq t$. Choose vertices $v_1 \in V_s$ and $w_1 \in V_t$ so that $\alpha(v) \leq \alpha(w_1) < \alpha(w_1) \leq \alpha(w)$ and there are no vertices $x \in V_s \cup V_t$ such that $\alpha(v_1) < \alpha(x) < \alpha(w_1)$. Then we have $\sigma^+(v) \geq \sigma^+(v_1)$ and $\sigma^+(w_1) \geq \sigma^+(w)$ by (1), and so $\Psi\Psi(\alpha, v) \geq \Psi(\alpha, v_1)$ and $\Psi(\alpha, w_1) \geq \Psi(\alpha, w)$. If $\alpha(v_1) = k$ and $\alpha(w_1) = k + m$, then we obtain by Lemma 2 that

$$0 \leq ||B(\alpha^k_m)|| - ||B(\alpha)|| = m(\Psi(\alpha, v_1) - \Psi(\alpha, w_1)).$$

Hence $\Psi(\alpha, v) \geq \Psi(\alpha, w)$, a contradiction. Consequently (2) is proved.

We next prove the sufficiency. Let α be a ranking which satisfies the conditions (1) and (2), and β be an optimal ranking. Note that β also satisfies the conditions (1) and (2) since we proved the necessity. Suppose $\alpha|_U = [u_1, \ldots, u_t, y, \ldots]$ and $x \neq y$ for some $U \in \{V_1, \ldots, V_r\}$. Then $\sigma^+(x) = \sigma^+(y)$, and we difine a ranking $\alpha\,'$ of D by

$$\alpha^{'}(u) = \begin{cases} \alpha(y) & \text{if } u = x \\ \alpha(x) & \text{if } u = y. \\ \alpha(u) & \text{otherwise} \end{cases}$$

It is clear that $\alpha^{'}|_U = [u_1, \ldots, u_t, y, \ldots]$, and it follows from Lemma 2 that $||B(\alpha^{'})|| = ||B(\alpha)||$. By repeating this procedure, we can get a ranking γ such that $||B(\gamma)|| = ||B(\alpha)||$, $\gamma|_U = \beta|_U$ for all $U \in \{V_1, \ldots, V_r\}$, and γ satisfies the conditions (1) and (2). It is easy to see that for any vertices v, $w \in V_i$, $1 \leq i \leq r$, we have $\Psi(\gamma, v) = \Psi(\beta, v)$, and $\Psi(\gamma, v) \neq \Psi(\gamma, w)$ if $v \neq w$. Therefore, we obtain $||B(\gamma)|| = ||B(\beta)||$ by (2) of Lemma 2, and conclude that α is an optimal ranking. $\square$

An r-partite complete PCD D with partite sets $V_1, \ldots, V_r$ is called a complete PCD if $|V_i| = 1$ for all i, $1 \leq i \leq r$. Then any two vertices of a complete PCD are joined by at least one arc.

Corollary[2]. Let D be a complete PCD and $\alpha = [v_1, \ldots, v_r]$ be a ranking of D. Then α is an optimal ranking if and only if

$$\sigma^+(v_1) \geq \cdots \geq \sigma^+(v_n).$$

Proof. Since $\Psi(\alpha, v) = \sigma^+(v)$ for all $v \in V(D)$, the corollary follows immediately from Theorem 1. $\square$

When we want ro rank the teams $V_1, \ldots, V_r$ instead of player $v_1, \ldots, v_n$, we can rank the teams as follows by using the corollary mentioned above. Let D be an r-partite complete PCD with partite sets $V_1, \ldots, V_r$. We first construct a complete PCD D^* with vertex set $V(D^*) = \{V_1, \ldots, V_r\}$ in which the weight of each arc V_iV_j is given by

$$\varepsilon(V_iV_j) = \frac{1}{|V_i||V_j|} \sum \varepsilon(uw)$$

where the summation is over all $uw \in A(D)$ such that $u \in V_i$ and $w \in V_j$. Applying the corollary to D^*, we can obtain all the optimal rankings of D^*.

It is easy to see that a semicomplete PCD(see[2]) is an r-partite complete PCD each of whose partite sets consists of one vertex or two vertices. Hence, the theorem in [2], by which we can get all the optimal ranking of a semicomplete PCD, is also a corollary of Theorem 1.

3 An r-partite PCD

Let D be an r-partite PCD with partite sets $V_1, \ldots, V_r$. An unordered pair $\{v, w\}$ of vertices of D is called an uncompared pair of D if v and w are not joined by arcs an if v and w are contained in distinct partite sets. Let $U^*(D)$ denote the set of all uncompared pairs of D. An r-partite complete PCD obtained from D by adding exactly one of arcs vw and wv for every uncompared pair $\{v, w\}$ of D is called an r-partite completion of D(which is called an r-partite completeness of D in [2]). It is clear that if $|U^*(D)| = t$, then there exist 2^t r-partite completion of D, and we denote by $C^*(D)$ the set of all r-partite completion of D. For example, let D be a 3-partite PCD given in Fig. 2. Then $U^*(D) = \{\{a_2, b_1\}, \{a_1, c_1\}\}$ and

$$C^*(D) = \{D + a_2b_1 + a_1c_1, D + a_2b_1 + c_1a_1, D + b_1a_2 + a_1c_1, D + b_1a_2 + c_1a_1\}.$$

In this section we shall $|U^*(D)|$ small. In order to do so, we need the next theorem.

Theorem 2 *Let D ba an r-partite complete PCD with n vertices, and let α be a ranking of D. Then*

$$||B(\alpha)|| = \sum_{v\in V(D)} \Psi(\alpha, v)\alpha(v) - \frac{1}{6}n(n^2-1).$$

Proof. We begin with a new notation. A function $\bar{\varepsilon} : V(D)\times V(D) \mapsto [0,1]$ is defined by

$$\bar{\varepsilon} = \begin{cases} \varepsilon(vw) & \text{if } vw \in A(D), \\ 0 & \text{otherwise.} \end{cases}$$

It is trivial that

$$\sigma_D^+(v) = \sum_{x\in V(D)} \bar{\varepsilon}(vx).$$

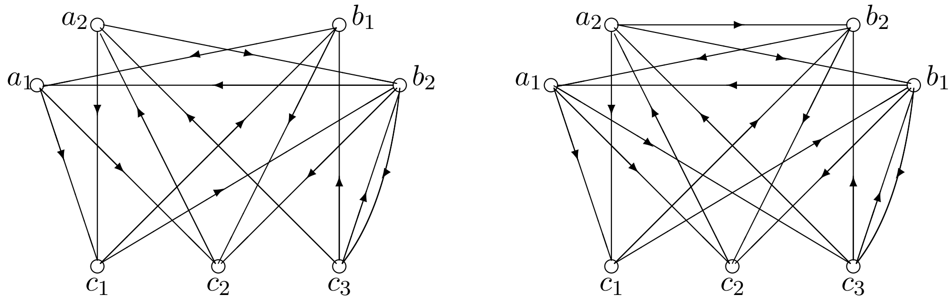

Fig.2. A 3-partite complete PCD D and$D + a_2b_1 + a_1c_1$.
The weight of each arc is 1 or 0.5.

We prove the theorem by induction on n. If $n = 1$ or 2, then the theorem holds at once. Suppose that the equation holds for $n = k$, and let $n = k+1 \geq 3$. Let w be the vertex such that $\alpha(w) = n$, and assume that $w \in V_t$. Put $U = V_t/\{w\}$, $V(D) = X \cup U \cup \{w\}$(disjoint union) and $H = D - w$, which is a PCD obtained from D by deleting w together with its incident arcs. Note that $V(H) = X \cup U$. By the inductive hypothesis, we have

$$\begin{aligned} ||B_D(\alpha)|| &= ||B_H(\alpha)|| + \sum_{v\in V(D)} \bar{\varepsilon}(wv)(n-\alpha(v)) \\ &= \sum_{v\in V(H)} \Psi_H(\alpha, v)\alpha(v) - \frac{1}{6}k(k^2-1) + \sum_{x_i n X} \bar{\varepsilon})(wx)(n-\alpha(x)). \end{aligned}$$

Since $\bar{\varepsilon}(wx) + \bar{\varepsilon}(xw) = 1$ for all $x \in X$, we obtain

$$\begin{aligned} \sum_{x\in X} \bar{\varepsilon}(wx)(n-\alpha(x)) &= n\sum_{x\in X} \bar{\varepsilon}(wx) + \sum x \in X(\bar{\varepsilon}(xw)-1)\alpha(x) \\ &= n\sigma_D^+(w) + \sum_{x\in X} \bar{\varepsilon}(xw)\alpha(x) - \sum_{x_i n X} \alpha(x). \end{aligned}$$

Hence

$$\begin{aligned}
||B_D(\alpha)|| =& \sum_{x\in X}\Psi_D(\alpha,x)\alpha(x) - \sum_{x\in X}\bar{\varepsilon}(xw)\alpha(x) + \sum_{u\in U}\Psi_D(\alpha,u)\alpha(u) \\
& - \sum_{u\in U}\alpha(u) - \frac{1}{6}k(k^2-1) + n\sigma_D^+(w) + \sum_{x\in X}\bar{\varepsilon}(xw)\alpha(x) - \sum_{x\in X}\alpha(x) \\
=& \sum_{v\in V(D)} \Psi_D(\alpha,v)\alpha(v) - \frac{1}{6}k(k^2-1) - \frac{1}{2}k(k+1) \\
& (\text{by } \Psi_D(\alpha,w) = n\sigma_D^+(w) \text{ and } \sum_{u\in U}\alpha(u) + \sum_{x\in X}\alpha(x) = \frac{1}{2}k(k+1)) \\
=& \sum_{v}\Psi_D(\alpha,v)\alpha(v) - \frac{1}{2}k(k+1) \\
=& \sum_{v}\Psi_D(\alpha,v)\alpha(v) - \frac{1}{6}n(n^2-1). \quad \square
\end{aligned}$$

For an r-partite PCD D, we denote by $l(D)$ the backward length of an optimal ranking of D. Namely, $l(D) = ||B(\alpha)||$ for $\alpha \in OR(D)$.

Lemma 3[2]. Let D be an r-partite PCD, and let $C^*(D) = \{D_1, \ldots, D_t\}$. Then $l(D) = \min\{l(D_1), \ldots, l(D_t)\}$. Moreover, if $\{D_i \in C^*(D) | l(D_i) = l(D)\} = \{D_a, \ldots, D_c\}$, then

$$OR(D) = OR(D_a) \cup \cdots \cup OR(D_c) \quad \text{(disjoint union)}.$$

By Lemma 3 and Theorems 1 and 2, we can easily obtain all the optimal rankings of an r-partite PCD D if D has a small number of uncompared pairs. For example, let D be a 3-partite PCD given in Fig. 2. Then

$$\begin{aligned}
C^*(D) = \{&D_1 = D + a_2b_1 + a_1c_1, \quad D_2 = D + a_2b_1 + c_1a_1. \\
&D_3 = D + b_1a_2 + a_1c_1, D_4 = D + b_1a_2 + c_1a_1\}.
\end{aligned}$$

It follows from Theorems 1 and 2 that

$$\begin{aligned}
&\alpha_1 = [c_1, a_1, b_2, a_2, c_3, b_1, c_2] \in OR(D_1), \\
&l(D_1) = ||B(\alpha_1)|| = 4.5 + 4\times 2 + 3.5\times 3 + 3\times 4 + 3\times 5 \\
&\qquad\qquad + 2\times 6 + 1\times 7 - 7(7^2-1)/6 = 13, \\
&\alpha_2 = [c_1, a_2, b_2, c_3, a_1, b_1, c_2] \in OR(D_2), \quad l(D_2) = 9, \\
&\alpha_3 = [c_1, a_1, b_1, c_3, b_2, a_2, c_2] \in OR(D_3), \quad l(D_3) = 12, \\
&\alpha_4 = [c_1, b_1, a_1, c_3, b_2, a_2, c_2] \in OR(D_4) \text{ and } l(D_4) = 10.
\end{aligned}$$

Hence, by Lemma 3, we have

$$OR(D) = OR(D_2) = \{\alpha = [c_1, a_2, b_2, c_3, u, w, c_2] | \{u, w\} = \{a_1, b_1\}\}.$$

We conclude this section with a remark on forward optimal rankings. Let α be a ranking of a PCD D. An arc vw of D is called a forward arc of α if $\alpha(v) < \alpha(w)$. We write $F(\alpha)$ for the set of all forward arcs of α,and define the forward length $||F(\alpha)||$ od α by

$$||F(\alpha)|| = \sum_{vw \in F(\alpha)} \varepsilon(vw)(\alpha(w) - \alpha(v)).$$

We say that α is a forward optimal ranking of D if $||F(\alpha)||$ is maximum(not minimum). Some results on forward optimal rankings can be found in [1]. For a ranking α of an r-partite PCD D with partite sets $V_1, \ldots, V_r$, we define a function $\Phi(\alpha, v)$ on $V(D)$ by

$$\Phi(\alpha, v) = \sigma^-(v) + |V_t| - \alpha|_{V_t}(v),$$

where $v \in V_t$ for all $v \in V(D)$. Then the following lemma holds, which can be proved as Theorem 2.

Lemma 4. Let D be an r-partite complete PCD with n vertices, and let αbe a ranking of D. Then

$$||F(\alpha)|| = \sum_{v \in V(D)} \Psi(\alpha, v)\alpha(v) - \frac{1}{6}n(n^2 - 1).$$

Note that it seems to be difficult to characterize forward optimal rankings of an r-partite complete PCD.

References

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[2] M. Kano and A. Sakamto, Ranking the vertices of a paired comparison digraph, *SIAM Algebraic Discrete Methods*,**6** (1985), 79–92.