Factors of Regular Graphs

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Abstract

A spanning subgraph F of a graph G is called a $[k-1,k]$-factor if $k-1 \leq d_F(x) \leq k$ for all vertices x of G, where $d_F(x)$ denotes the degree of x in F. Tutte proved that if r is an odd integer, then every r-regular graph has a $[k-1,k]$-factor for every integer k, $0 < k < r$. We prove that if r is odd and $0 < k \leq \frac{2r}{3}$, then every r-regular graph has a $[k-1,k]$-factor each of whose components is regular.

1 Introduction

We consider a finite graph G which may have multiple edges but has no loops. A graph without multiple edges is called a *simple graph*. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G, respectively. We write $d_G(x)$ for the degree of a vertex x in G. Let a, b and r be integers such that $0 \leq a \leq b$ and $r > 0$. A spanning subgraph F of G is called an $[a,b]$-factor of G if $a \leq d_F(x) \leq b$ for all $x \in V(G)$, and we usually call an $[r,r]$-factor an *r-factor*. An r-*regular graph* is a graph in which each vertex has degree r. Other notation and definitions not defined in this paper can be found in [3 or 4].

Tutte [12]([4,p77]) proved that for any odd integer r and any integer k $(0 < k < r)$, every r-regular graph has a $[k-1,k]$-factor. It was proved in [6,11] that every regular graph has a $[1,2]$-factor each of whose components is regular. Enomoto and Saito [5] gave the following conjecture: Every r-regular graph has a $[k-1,k]$-factor each of whose components is regular for any k, $0 < k < r$. Note that this conjecture is true when r is even by Petersen's 2-factorable theorem (see Lemma **??**). So the essential part of the conjecture is the case that r is odd. Main results of this paper are the following two theorems, and a theorem on $[a,b]$-factors is given in Section 3.

Theorem 1. *Let r and k be positive integers. If $k \leq 2(2r+1)/3$, then every $(2r+1)$-regular graph has a $[k-1,k]$-factor each of whose components is regular.*

Theorem 2. *Let k and r be positive integers. If $2r + 2 - \sqrt{2r+2} < k \leq 2r$, then there exists a simple $(2r+1)$-regular graph that has no $[k-1, k]$-factor each of whose components is regular.*

2 Proofs of Theorem 1 and 2

Let G be a graph, and g and f be integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. A spanning subgraph F of G is called a (g, f)- *factor* if $g(x) \leq d_F(x) \leq f(x)$ for all $x \in V(G)$. A (g, f)-factor satisfying $g(x) = f(x)$ for all $x \in V(G)$ is briefly called an *f-factor*. For a vertex subset X of G, we write $G - X$ for the subgraph of G obtained from G by deleting the vertices in X together with their incident edges.

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