Almost-Regular Factorization of Graphs

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Abstract

For integers a and b, $0 \leq a \leq b$, an $[a,b]$-graph G satisfies $a \leq \deg(x,G) \leq b$ for every vertex x of G, and an $[a,b]$-factor is a spanning subgraph F such that $a \leq \deg(x,F) \leq b$ for every vertex x of F. An $[a,b]$-factor is almost-regular if $b = a+1$. A graph is $[a,b]$-factorable if its edges can be decomposed into $[a,b]$-factors. When both k and t are positive integers and s is a nonnegative integer we prove that every $[(12k+2)t+2ks,(12k+4)t+2ks]$-graph is $[2k,2k+1]$-factorable. As its corollary, we prove that every $[r,r+1]$-graph with $r \geq 12k^2+2k$ is $[2k,2k+1]$-factorable, which is a partial extension of the two results, one by Thomassen and the other by Era.

1 Introduction

We deal with finite graphs which may have multiple edges but no loops. All notation and definitions not given here can be found [1] or [3].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex x of a subgraph H of G, we denote the degree of x in H by $\deg(x,H)$. For integers a and b, $0 \leq a \leq b$, an $[a,b]$-graph G satisfies $a \leq \deg(x,G) \leq b$ for every $x \in V(G)$ and an an $[a,b]$-factor of a graph is a spanning subgraph F such that $a \leq \deg(x,F) \leq b$ for every vertex x of G. If $E(G)$ can be partitioned into $[a,b]$-factor $F_1,\ldots,F_n$ of G, then their union $F_1 \cup \cdots \cup F_n$ is called an $[a,b]$-factorization of G and G is said to be $[a,b]$-factorable. When G has an $[a,b]$-factorization $F_1 \cup \cdots \cup F_n$, we briefly say that G is decomposed into $[a,b]$-factors $F_1,\ldots,F_n$. We usually call an $[r,r]$-graph an r-regular graph. Similarly, an $[r,r]$-factor, an $[r,r]$-factorable graph are called an r-factor, r-factorization and an r-factorable graph, respectively. Our main result is an follows:

Theorem. For any positive integers k and t, and a nonnegative integers s, every $[(12k+2)t+2ks, (12k+4)t+2ks]$-graph is $[2k, 2k+1]$-factorable.

2 Proof of theorem

We begin with a definition. Let G be a graph and g and f be integer-valued functions defined on $V(G)$ such that $g(x) \le f(x)$ for all $x \in V(G)$. Then a (g,f)-factor of G is a spanning subgraph F satisfying $g(x) \le \deg(x,F) \le f(x)$ for all $x \in V(G)$. We apply the following five lemmas to the proof of our theorem.

Lemma A (Petersen[6], [3, Theorem 9.9]). Let G be a graph which may have loops. Then G is 2-factorable if and only if G is a $2n$-regular graph for some positive integer n.

Lemma B (Kano[4]). Let a and b integers, $0 \le a \le b$. Then a graph G is $[2a, 2b]$-factorable if and only if G is a $[2an, 2bn]$-graph for some positive integer n.

Lemma C (Kano[4]). Let G be an n-edge-connected graph($n \ge 1$) and θ be a real number such that $0 \le \theta \le 1$. Let g and f be two integer-valued functions defined on $V(G)$ which satisfy the following inequalities:

(1) $g(x) \le \theta \deg(x,G) \le f(x)$ for all $x \in V(G)$.

If G satisfies the following condition (2) and either one of (3a) or (3b), then G has a (g,f)-factor:

(2) G has at least one vertex v with $g(v) < f(v)$, or $g(x) = f(x)$ for all $x \in V(G)$ and $\sum f(x)$ over all x in $V(G)$ is even.

(3a) Both sets $\{\deg(x,G) | g(x) = f(x), x \in V(G)\}$ and $\{f(x) | g(x) = f(x), x \in V(G)\}$ contain only even numbers.

(3b) $\{\deg(x,G) | g(x) = f(x), x \in V(G)\}$ consists of even numbers, n is odd, $(n+1)\theta \ge 1$ and $(n+1)(1-\theta) \ge 1$.

Lemma 1. Every $[6k, 6k+2]$-graph with at most one vertex of degree $6k$ can be decomposed into three $[2k, 2k+1]$-factors.

Proof. Let G be a $[6k, 6k+2]$-graph with at most one vertex of degree $6k$. We prove only the case that G has a vertex of degree $6k$, since in the other case the proof is quite similar. Let u be the vertex of degree $6k$ in G. We obtain a $[6k+1, 6k+2]$-graph G_1 from G by adding a loop to G at u. Then, by Theorem 1.2 in [1], there exists a $(6k+2)$-regular supergraph G_2 of G_1. By Lemma A, G_2 can be decomposed into $3k+1$ 2-factors consisting of $\bar{F}_0$ and the $\bar{M}_i (1 \le i \le 3k)$, where we may assume that $\bar{F}_0$ contains the loop. Put

$$\bar{F}_1 = \cup_{i=1}^{k} M_i, \quad \bar{F}_2 = \cup_{i=k+1}^{2k} M_i, \quad F_3 = \cup_{i=2k+1}^{3k} M_i.$$

Now, for $i = 0$ to 3, Let F_i be the spanning subgraph of G whose edge set is $E(G) \cap E(\bar{F}_i)$. Then F_0 is a $[0,2]$-factor and each of F_1, F_2, and F_3 is a $[2k-1, 2k]$-factor of G. Note that for each vertex x of G, there exists at most one F_i, $1 \le i \le 3$. We shall develop a procedure for this purpose in the following two steps.

Step 1. Let W be the set of vertices x of G for which there exists some F_i, $1 \le i \le 3$, such that $\deg(x, F_i) = 2k-2$. We denote by K a subgraph of F_0 induced by the edge set $S = \{e \in E(F_0) | e$ is incident with a vertex of $W\}$. Then it is clear that $\deg(x, S) = 2$ for every vertex $x \in W$ and each component of K is either a path or a cycle. We orient all the edges of K so that each component of the resulting digraph $D(K)$ is either a directed path or a directed cycle.

Procedure 1. For every vertex x of W, there exists exactly one F_i, $1 \le i \le 3$, such that $\deg(x, F_i) = 2k-1$. Add to F_i the edge of S which is adjacent to x in the digraph $D(K)$.

After applying procedure 1 to all the vertices of W, we obtain three supergraph H_i of $F_i (i = 1, 2, 3)$. It is easily verified that each H_i is a $[2k, 2k+1]$-factor of G. We now proceed to the next step.

Step 2. Let L be the set of edges of F_0 which are contained in none of $H_i (i = 1, 2, 3)$. Then we have that

(i) $\deg(xL) = 2$ implies $\deg(x, G) = 6k+2$ and $\deg(x, H_i) = 2k$ for all i.

(ii) $\deg(x, L) = 1$, $\deg(x, G) = 6k+2$ implies $\deg(x, H_i) = 2k$ for two $i, 1 \le i \le 2$.

(iii) $\deg(x, L) = 1$, $\deg(x, G) = 6k+1$ implies $\deg(x, H_i) = 2k$ for all i, $1 \le i \le 2$.

Together with this observation and the fact that each component of L is either a path or a cycle, we can continue the next operation on the algorithm.

Procedure II. Add all the edges of L to some suitable $H_i (1 \le i \le 3)$ so that the resulting graphs are still $[2k, 2k+1]$-factors of G.

After doing this, we obtain a $[2k, 2k+1]$-factorization of G. □

Lemma 2. For each positive integer k, every $[12k+2, 12k+4]$-graph can be decomposed into six $[2k, 2k+1]$-factors.

Proof. Without loss of generality, we may assume that a given $[12k+2, 12k+4]$-graph G is connected. We prove only the case that G has a vertex of degree $12k+2$ in G. Set $\theta = 1/2$ and define two integer-valued functions g and f on $V(G)$ by

$$g(x) = \begin{cases} 6k & \text{for } x = w; \\ 6k+1 & \text{for } x \neq w \text{ with } \deg(x,G) \leq 12k+3 \\ 6k+2 & \text{when } \deg(x,G) = 12k+4, \end{cases}$$

and

$$f(x) = \begin{cases} 6k+1 & \text{when } \deg(x,G) = 12k+2 \\ 6k+2 & \text{when } \deg(x,G) \geq 12k+3. \end{cases}$$

Then θ, $g(x)$, $f(x)$ and $n = 1$ satisfy the conditions (1), (2), and (3b) of Lemma C. Hence G has a (g, f)-factor, that is, it has a $[6k, 6k+2]$-factor F_1 containing at most one vertex of degree $6k$. On the other hand, the remaining subgraph $F_2 = G - E(F_1)$ is a $[6k+1, 6k+2]$-factor of G. Applying Lemma 2 to F_1 and F_2, both F_1 and F_2 can be decomposed into three $[2k, 2k+1]$-factors. Therefore the lemma is proved. □

Proof of Theorem. Let G be a $[(12k+2)t + 2ks, (12k+4)t + 2ks]$-graph. The proof is by induction on S. We first show that the theorem is true when $s = 0$. It follows from Lemma B and Lemma 2 that every $[(12k+2)t, (12k+4)t]$-graph can be decomposed into $6t[2k, 2k+1]$-factors.

Assume that $s \geq 1$ and G is a connected $[(12k+2)t+2ks, (12k+4)t+2ks]$-graph. Set $\theta = (2k)/((12k+2)t + 2ks)$ and now define two functions g and f on $V(G)$ by

$$g(x) = 2k \quad \text{for all} \quad x \in V(G),$$

Then θ, $g(x), f(x)$, $n = 1$ satisfy the conditions (1), (2), and (3a) of Lemma C. Hence G has a $[2k, 2k+1]$-factor F such that the remaining subgraph $G - E(F)$ is a $[(12k+2)t + 2k(s-1), (12k+4) + 2k(s-1)]$-graph. Therefore, by induction hypothesis, the proof is complete. □

Lovász[5] and Tutte[8] showed that every r-regular graph has a $[k, k+1]$-factor for all k, $1 \leq k \leq r-1$. Then Thomassen[7] gave a slight generalization of this result for all k, $1 \leq k \leq r$. Moreover, Era[2] recently proved that every r-regular graph is $k, k+1]$-factorable when $r \geq 2k^2$. We now show that their results in the case of even k is a special case of theorem.

Corollary. The following statements hold:

(1) If $r \geq 12k^2 + 2k$, then every $[r, r+1]$-graph is $[2k, 2k+1]$-factorable.

(2) Let t be an integer greater than one. If $r \geq (12k^2 + 2k)t$, then every $[r, r+2(t-1)k]$-graph is $[2k, 2k+1]$-factorable.

Proof. We shall prove only (2) since (1) can be proved similarly. Let $t \geq 2$, $r \geq (12k^2 + 2k)t$. Then there exist integers m and n satisfying

$r = (12k+2)m + n$, $0 \le n < 12k+2$, and $m \le kt$. For the integer n, there exists integers c and d such that $n = 2kc + d$, $0 \le c$ and $0 \le d < 2k$. Then we have the following inequalities:

$$(12k+2)m + 2kc \le (12k * 2)m + n = r,$$

and

$$\begin{aligned} r + 2(t-1)k &\le (12k+2)m + 2kc + 2k + 2(t-1)k \\ &\le (12k+4)m + 2kc. \end{aligned}$$

Therefore an $[r, r+2(t-1)k]$-graph is a $[(12k+2)m+2kc, (12k+4)m+2kc]$-graph. Consequently, the corollary follows from the theorem. □

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