Almost-Regular Factorization of Graphs

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Abstract

For integers $a$ and $b$, $0 \leq a \leq b$, an $[a, b]$-graph $G$ satisfies $a \leq \deg(x, G) \leq b$ for every vertex $x$ of $G$, and an $[a, b]$-factor is a spanning subgraph $F$ such that $a \leq \deg(x, F) \leq b$ for every vertex $x$ of $F$. An $[a, b]$-factor is almost-regular if $b = a + 1$. A graph is $[a, b]$-factorable if its edges can be decomposed into $[a, b]$-factors. When both $k$ and $t$ are positive integers and $s$ is a nonnegative integer we prove that every $[(12k + 2)t + 2ks, (12k + 4)t + 2ks]$-graph is $[2k, 2k + 1]$-factorable. As its corollary, we prove that every $[r, r + 1]$-graph with $r \geq 12k^2 + 2k$ is $[2k, 2k + 1]$-factorable, which is a partial extension of the two results, one by Thomassen and the other by Era.

1 Introduction

We deal with finite graphs which may have multiple edges but no loops. All notation and definitions not given here can be found \cite{1} or \cite{3}.

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $x$ of a subgraph $H$ of $G$, we denote the degree of $x$ in $H$ by $\deg(x, H)$. For integers $a$ and $b$, $0 \leq a \leq b$, an $[a, b]$-graph $G$ satisfies $a \leq \deg(x, G) \leq b$ for every $x \in V(G)$ and an $[a, b]$-factor of a graph is a spanning subgraph $F$ such that $a \leq \deg(x, F) \leq b$ for every vertex $x$ of $G$. If $E(G)$ can be partitioned into $[a, b]$-factor $F_1, \ldots, F_n$ of $G$, then their union $F_1 \cup \cdots \cup F_n$ is called an $[a, b]$-factorization of $G$ and $G$ is said to be $[a, b]$-factorable. When $G$ has an $[a, b]$-factorization $F_1 \cup \cdots \cup F_n$, we briefly say that $G$ is decomposed into $[a, b]$-factors $F_1, \ldots, F_n$. We usually call an $[r, r]$-graph an $r$-regular graph. Similarly, an $[r, r]$-factor, an $[r, r]$-factorable graph are called an $r$-factor, $r$-factorization and an $r$-factorable graph, respectively. Our main result is as follows:
Theorem. For any positive integers $k$ and $t$, and a nonnegative integers $s$, every $[(12k + 2)t + 2ks, (12k + 4)t + 2ks]$-graph is $[2k, 2k + 1]$-factorable.

2 Proof of theorem

We begin with a definition. Let $G$ be a graph and $g$ and $f$ be integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then a $(g, f)$-factor of $G$ is a spanning subgraph $F$ satisfying $g(x) \leq \deg(x, F) \leq f(x)$ for all $x \in V(G)$. We apply the following five lemmas to the proof of our theorem.

Lemma A (Petersen[6], [3, Theorem 9.9]). Let $G$ be a graph which may have loops. Then $G$ is 2-factorable if and only if $G$ is a $2n$-regular graph for some positive integer $n$.

Lemma B (Kano[4]). Let $a$ and $b$ integers, $0 \leq a \leq b$. Then a graph $G$ is $[2a, 2b]$-factorable if and only if $G$ is a $[2an, 2bn]$-graph for some positive integer $n$.

Lemma C (Kano[4]). Let $G$ be an $n$-edge-connected graph $(n \geq 1)$ and $\theta$ be a real number such that $0 \leq \theta \leq 1$. Let $g$ and $f$ be two integer-valued functions defined on $V(G)$ which satisfy the following inequalities:

\[ (1) \quad g(x) \leq \theta \deg(x, G) \leq f(x) \quad \text{for all} \quad x \in V(G). \]

If $G$ satisfies the following condition (2) and either one of (3a) or (3b), then $G$ has a $(g, f)$-factor:

\[ (2) \quad G \text{ has at least one vertex } v \text{ with } g(v) < f(v), \text{ or } g(x) = f(x) \text{ for all } x \in V(G) \text{ and } \sum f(x) \text{ over all } x \text{ in } V(G) \text{ is even.} \]

\[ (3a) \quad \text{Both sets } \{\deg(x, G) | g(x) = f(x), x \in V(G)\} \text{ and } \{f(x) | g(x) = f(x), x \in V(G)\} \text{ contain only even numbers.} \]

\[ (3b) \quad \{\deg(x, G) | g(x) = f(x), x \in V(G)\} \text{ consists of even numbers, } n \text{ is odd, } (n + 1)\theta \geq 1 \text{ and } (n + 1)(1 - \theta) \geq 1. \]

Lemma 1. Every $[6k, 6k + 2]$-graph with at most one vertex of degree $6k$ can be decomposed into three $[2k, 2k + 1]$-factors.

\[ \text{Proof.} \quad \text{Let } G \text{ be a } [6k, 6k + 2]\text{-graph with at most one vertex of degree } 6k. \text{ We prove only the case that } G \text{ has a vertex of degree } 6k, \text{ since in the other case the proof is quite similar. Let } u \text{ be the vertex of degree } 6k \text{ in } G. \text{ We obtain a } [6k + 1, 6k + 2]\text{-graph } G_1 \text{ from } G \text{ by adding a loop to } G \text{ at } u. \text{ Then, by Theorem 1.2 in [1], there exists a } (6k + 2)\text{-regular supergraph } G_2 \text{ of } G_1. \text{ By Lemma A, } G_2 \text{ can be decomposed into } 3k + 1 \text{ } 2\text{-factors consisting of } \bar{F}_0 \text{ and the } \bar{M}_i(1 \leq i \leq 3k), \text{ where we may assume that } \bar{F}_0 \text{ contains the} \]
loop. Put

\[ F_1 = \bigcup_{i=1}^{k} M_i, \quad F_2 = \bigcup_{i=k+1}^{2k} M_i, \quad F_3 = \bigcup_{i=2k+1}^{3k} M_i. \]

Now, for \( i = 0 \) to 3, let \( F_i \) be the spanning subgraph of \( G \) whose edge set is \( E(G) \cap E(\bar{F}_i) \). Then \( F_0 \) is a \([0, 2]\)-factor and each of \( F_1, F_2, \) and \( F_3 \) is a \([2k-1, 2k]\)-factor of \( G \). Note that for each vertex \( x \) of \( G \), there exists at most one \( F_i, 1 \leq i \leq 3 \). We shall develop a procedure for this purpose in the following two steps.

**Step 1.** Let \( W \) be the set of vertices \( x \) of \( G \) for which there exists some \( F_i, 1 \leq i \leq 3 \), such that \( \deg(x, F_i) = 2k - 2 \). We denote by \( K \) a subgraph of \( F_0 \) induced by the edge set \( S = \{ e \in E(F_0) | e \) is incident with a vertex of \( W \} \). Then it is clear that \( \deg(x, S) = 2 \) for every vertex \( x \in W \) and each component of \( K \) is either a path or a cycle. We orient all the edges of \( K \) so that each component of the resulting digraph \( D(K) \) is either a directed path or a directed cycle.

**Procedure 1.** For every vertex \( x \) of \( W \), there exists exactly one \( F_i, 1 \leq i \leq 3 \), such that \( \deg(x, F_i) = 2k - 1 \). Add to \( F_i \) the edge of \( S \) which is adjacent to \( x \) in the digraph \( D(K) \).

After applying procedure 1 to all the vertices of \( W \), we obtain three supergraph \( H_i \) of \( F_i (i = 1, 2, 3) \). It is easily verified that each \( H_i \) is a \([2k, 2k + 1]\)-factor of \( G \). We now proceed to the next step.

**Step 2.** Let \( L \) be the set of edges of \( F_0 \) which are contained in none of \( H_i (i = 1, 2, 3) \). It is easily verified that each \( H_i \) is a \([2k, 2k + 1]\)-factor of \( G \). We now proceed to the next step.

**Procedure II.** Add all the edges of \( L \) to some suitable \( H_i (1 \leq i \leq 3) \) so that the resulting graphs are still \([2k, 2k + 1]\)-factors of \( G \).

After doing this, we obtain a \([2k, 2k + 1]\)-factorization of \( G \).

**Lemma 2.** For each positive integer \( k \), every \([12k + 2, 12k + 4]\)-graph can be decomposed into six \([2k, 2k + 1]\)-factors.

**Proof.** Without loss of generality, we may assume that a given \([12k + 2, 12k + 4]\)-graph \( G \) is connected. We prove only the case that \( G \) has a vertex
of degree $12k + 2$ in $G$. Set $\theta = 1/2$ and define two integer-valued functions $g$ and $f$ on $V(G)$ by

$$g(x) = \begin{cases} 
6k & \text{for } x = w; \\
6k + 1 & \text{for } x \neq w \text{ with } \deg(x, G) \leq 12k + 3 \\
6k + 2 & \text{when } \deg(x, G) = 12k + 4,
\end{cases}$$

and

$$f(x) = \begin{cases} 
6k + 1 & \text{when } \deg(x, G) = 12k + 2 \\
6k + 2 & \text{when } \deg(x, G) \geq 12k + 3.
\end{cases}$$

Then $\theta, g(x), f(x)$ and $n = 1$ satisfy the conditions (1), (2), and (3b) of Lemma C. Hence $G$ has a $(g, f)$-factor, that is, it has a $[6k, 6k + 2]$-factor $F_1$ containing at most one vertex of degree $6k$. On the other hand, the remaining subgraph $F_2 = G - E(F_1)$ is a $[6k + 1, 6k + 2]$-factor of $G$. Applying Lemma 2 to $F_1$ and $F_2$, both $F_1$ and $F_2$ can be decomposed into three $[2k, 2k + 1]$-factors. Therefore the lemma is proved.

**Proof of Theorem.** Let $G$ be a $[(12k + 2)t + 2ks, (12k + 4)t + 2ks]$-graph. The proof is by induction on $S$. We first show that the theorem is true when $s = 0$. It follows from Lemma B and Lemma 2 that every $[(12k + 2)t, (12k + 4)t]$-graph can be decomposed into $6t[2k, 2k + 1]$-factors.

Assume that $s \geq 1$ and $G$ is a connected $[(12k + 2)t + 2ks, (12k + 4)t + 2ks]$-graph. Set $\theta = (2k)/((12k + 2)t + 2ks)$ and now define two functions $g$ and $f$ on $V(G)$ by

$$g(x) = 2k \text{ for all } x \in V(G),$$

Then $\theta, g(x), f(x), n = 1$ satisfy the conditions (1), (2), and (3a) of Lemma C. Hence $G$ has a $[2k, 2k + 1]$-factor $F$ such that the remaining subgraph $G - E(F)$ is a $[(12k + 2)t + 2k(s - 1), (12k + 4) + 2k(s - 1)]$-graph. Therefore, by induction hypothesis, the proof is complete. □

Lovašs[5] and Tutte[8] showed that every $r$-regular graph has a $[k, k + 1]$-factor for all $k, 1 \leq k \leq r - 1$. Then Thomassen[7] gave a slight generalization of this result for all $k, 1 \leq k \leq r$. Moreover, Era[2] recently proved that every $r$-regular graph is $k, k + 1$-factorable when $r \geq 2k^2$. We now show that their results in the case of even $k$ is a special case of theorem.

**Corollary.** The following statements hold:

1. If $r \geq 12k^2 + 2k$, then every $[r, r + 1]$-graph is $[2k, 2k + 1]$-factorable.
2. Let $t$ be an integer greater than one. If $r \geq (12k^2 + 2k)t$, then every $[r, r + 2(t - 1)k]$-graph is $[2k, 2k + 1]$-factorable.

**Proof.** We shall prove only (2) since (1) can be proved similarly. Let $t \geq 2$, $r \geq (12k^2 + 2k)t$. Then there exist integers $m$ and $n$ satisfying
\[ r = (12k + 2)m + n, \ 0 \leq n < 12k + 2, \text{ and } m \leq kt. \] For the integer \( n \), there exists integers \( c \) and \( d \) such that \( n = 2kc + d, \ 0 \leq c \) and \( 0 \leq d < 2k \). Then we have the following inequalities:

\[ (12k + 2)m + 2kc \leq (12k \cdot 2)m + n = r, \]

and

\[ r + 2(t - 1)k \leq (12k + 2)m + 2kc + 2k + 2(t - 1)k \leq (12k + 4)m + 2kc. \]

Therefore an \([r, r+2(t-1)k]\)-graph is a \([(12k+2)m+2kc, (12k+4)m+2kc]\)-graph. Consequently, the corollary follows from the theorem. \(\Box\)

**References**


