$[a, b]$-Factorization of a Graph

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Abstract

Let a and b integers such that $0 \le a \le b$. Then a graph G is called an $[a, b]$-graph if $a \le d_G(x) \le b$ for every $x \in V(G)$, and an $[a, b]$-factor of a graph is defined to be its spanning subgraph F such that $a \le d_G(x) \le b$ for every vertex x, where $d_G(x)$ and $d_F(x)$ denote the degrees of x in G and F, respectively. If the edges of a graph can be decomposed the following two theorems: (i) a graph G is $[2a, 2b]$-factorable if and only if G is a $[2am, 2bm]$-graph for some integer m, and (ii) every $[8m + 2k, 10m + 2]$-graph is $[1, 2]$-factorable.

1 Introduction

We deal with finite *graphs* which may have multiple edges but have no loops. A graph without multiple edges is called a *simple graph*. All notation and definitions not given here can be found in [4].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and H be a subgraph of G. For a vertex x of H, we denote the degree of x in H by $d_H(x)$, in particular, the degree of a vertex y of G is denoted by $d_G(y)$. Let a and b be integers such that $0 \le a \le b$. Then a graph G is called an $[a, b]$-graph if $a \le d_G(x) \le b$ for every $x \in V(G)$, and an $[a, b]$-*subgraph* can be defined similarly. A spanning $[a, b]$-subgraph is called an $[a, b]$-*factor*. Then, if F is an $[a, b]$-factor of a graph G, then $a \le d_F(x) \le b$ for all $x \in V(G)$.

If the edges of a graph G can be decomposed into $[a, b]$-factors $F_1, \cdots, F_n$ of G, then the union $F_1 \cup \cdots \cup F_n$ is called an $[a, b]$-*factorization* of G and G itself is to be $[a, b]$-*factorable*.

We usually call an $[r,r]$-graph an r-*regular graph*. Similarly, an $[r,r]$-factor, an [r,r]-factorization and an [r,r]-factorable graph are called an *r-factor*, an r-*factorization*, and an r-*factorable graph*, respectively.

In 1891 Petersen [4, Theorem 8.8 ,13] obtained the following theorem.

Theorem 1.1 *A graph G is 2 factorable if and only if G is a $2m$-regular graph for some positive integer m..*

Recently, Akiyama [1] proved that *every r-regular graph is $[2,3]$-factorable, where $r \geq 2$* This is the first contribution toward [a,b]-factorization with $a < b$. Era [6] proved that *if $r \geq 2k^2$, then r-regular simple graph is $[k,k+1]$-factorable.* We now give our theorems.

Theorem 1.2 *Let $0 \leq a \leq b$. Then a graph G is $[2a,2b]$-factorable if and only if G is a $[2am,2bm]$-graph for some positive integer m.*

The theorem is an extension of Theorem 1.1.

Theorem 1.3 *Let $m \geq 1$ and $k \geq 0$. Then every $[8m+2k, 10m+2k]$-graph is $[1,2]$-factorable.*

As a corollary of this theorem, we can obtain the next result.

Corollary 1.4 (1) *if $r \geq 8m$ and $m \geq 1$, then every $[r, r+2m-1]$-graph $[1,2]$-factorable.*

(2) *Every connected $[r,r+1]$-graph is $[1,2]$-factorable, where $r \leq 1$.*

Note that a $[2am, 2bm]$-graph can be decomposed into $[2a,2b]$-factors, and a $[8m+2k, 10m+2k]$-graph can be decomposed into $6m+k[1,2]$-factors. But the number of $[a,b]$-factors in an $[a,b]$-factorization of a graph is not uniquely determined. For example, a $4m$-regular graph can be decomposed into $k[1,2]$-factors for every k, $2m \leq k \leq 3m$ (see Theorem 1.1 and Lemma 4.1). It is clear that the union of an odd cycle and a cubic graph, which is a $[2,3]$-graph with two components, is not $[1,2]$-factorable. So the connectivity of a graph in (2) of Corollary 1.4 is necessary. Moreover, we show that there exists a $[6,8]$-graph which is not $[1,2]$-factorable (Remark 4.3).

We next mention two factor theorems on which our proof will heavily depend. One is Lovász's (g,f)-factor theorem (see Lemma 2.2), which plays an important role throughout this article, and the order is Theorem 2.1, which is proved by making use of Lovász's (g,f)-factor theorem. By Theorem 2.1, not only can we prove many known theorems on r-factors due to Baebler, Gallai, Petersen, and others, but also we can obtain some new results on $[a,b]$-factors, for instance, Theorem 1.2 is an easy consequence of it.

Let us finally note a survey article [2], in which many results related to our theorems can be found.

2 Factor Theorem

We begin by introducing some new notation and definitions. For a finite set X, we denote by $|X|$ the number of elements in X. Let G be a graph, and g and f be two integer-valued functions defined on $V(G)$. For a subset S of $V(G)$, we write $G-S$ for the subgraph of G obtained from G by deleting the vertices in S together with their incident edges. If S and T are disjoint subsets of $V(G)$, then $e(S,T)$ denotes the number of edges of G joining S and T.

In this section we shall prove the following theorem and give some its corollaries.

Theorem 2.1 *Let G be an n-edge connected graph ($n \geq 1$), θ be a real number such that $0 \leq \theta \leq 1$, and g and f be two integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. If one of* {(1a), (1b)}, (2) *and one of* {(3a), (3b), (3c), (3d), (3e), (3f)} *hold, then G has a (g,f)-factor*

(1a) *$g(x) \leq \theta d_G(x) \leq f(x)$ for all $x \in V(G)$.*

(1b) $\epsilon = \sum_{x\in V(G)}[\max\{0, g(x) - \theta d_G(x)\} + \max\{0, \theta d_G(x) - f(x)\}] < 1$.

(2) *G has at least one vertex v such that $g(v) < f(v)$; or $g(x) = f(x)$ for all $x \in V(G)$ and $\sum_{x\in V(G)} f(x) \equiv 0 \pmod 2$.*

(3a) *$n\theta \geq 1$ and $n(1-\theta) \geq 1$.*

(3b) *$\{d_G(x)|g(x) = f(x), x \in V(G)\}$ and $\{f(x)|g(x) = f(x), x \in V(G)\}$ both consist of even numbers.*

(3c) *$\{d_G(x)|g(x) = f(x), x \in V(G)\}$ consists of even numbers, n is odd, $(n+1)\theta \geq 1$, where $(n+1)\theta \geq 1$ and $(n+1)(1-\theta) \geq 1$.*

(3d) *$\{f(x)|g(x) = f(x), x \in V(G)\}$ consists of even numbers and $m(1-\theta) \geq 1$, where $m \in \{n, n+1\}$ and $m \equiv 1 \pmod 2$.*

(3e) *$\{d_G(x)|g(x) = f(x), x \in V(G)\}$ and $\{f(x)|g(x) = f(x), x \in V(G)\}$ both consist of odd numbers and $m\theta \geq 1$, where $m \in \{n, n+1\}$ and $m \equiv 1 \pmod 2$.*

(3f) *$g(x) < f(x)$ for every $x \in V(G)$* (see [8]).

Note that similar necessary conditions for a graph to have a (g,f)-factor which contains p given edges but has no q given edges are obtained in [9]. In order to prove the above theorem we need the next (g,f)-factor theorem due to Lovász, to which Tutte [16] gave a short proof.

Lemma 2.2 (Lovász [12], [16, Theorem 7.2]) *Let G be a graph and g and f be integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all*

$x \in V(G)$. Then G has a (g,f)-factor if and only if

$$\delta(S,T) = \sum_{t\in T}\{d_G(t) - g(t)\} + \sum_{s\in S} f(s) - e(S,T) - h(S,T) \tag{2.1}$$

for all disjoint S and T of $V(G)$, where $h(S,T)$ denotes the number of components C of $G-(S\cup T)$ such that $g(x) = f(x)$ for all $x \in V(C)$ and $e(T,V(C)) + \sum_{x\in V(G)} f(x) \equiv 1 \pmod 2$.

Note that the condition $0 \le g(x) \le f(x) \le d_G(x)$ in [12] and [16] can be replaced by $g(x) \le f(x)$ as above [10, 15].

Proof of Theorem 2.1. We shall prove that two functions g and f in Theorem 2.1 satisfy the condition (2.1) in Lemma 2.2. It is obvious that (1a) and (1b). Hence we may assume (1b) holds. Let $\{C_1,\cdots,C_r\}$ be the set of components of $G-(S\cup T)$ which satisfy the conditions on $h(S,T)$, where $r = h(S,T)$. By (1b) of Theorem 2.1, we have

$$\begin{aligned}
\delta(S,T) \ge{}& (1-\theta)\sum_{t\in T} d_G(t) + \theta\sum_{s\in S} d_G(s) - \sum_{t\in T}\max\{0, g(t)-\theta d_G(t)\} \\
&- \sum_{s\in S}\max\{0,\theta d_G(s) - f(s)\} - e(S,T) - r \qquad (2.2)\\
\ge{}& (1-\theta)\Big\{e(T,S) + \sum_{i=1}^{r} e(T,V(C_i))\Big\} \\
&+\theta\Big\{e(S,T) + \sum_{i=1}^{r} e(S,V(C_i))\Big\} \\
&- \epsilon - e(S,T) - r \\
={}& \sum_i \{(1-\theta)e(T,V(C_i)) + \theta e(S,V(C_i)) - 1\} - \epsilon. \qquad (2.3)
\end{aligned}$$

Since $\delta(S,T)$ is an integer and $\epsilon < 1$ it suffices to show that $\delta(S,T) \ge -\epsilon$.

If (3f) holds, then $r=0$ and so $\delta(S,T) \ge -\epsilon$. Hence we may assume that G satisfies (2) and one of {(3a),(3b),(3c),(3d),(3e)}. Take any $C \in \{C_1,\cdots,C_r\}$, and put

$$\Delta(C) = (1-\theta)e(T,V(C)) + \theta e(S,V(C)) - 1.$$

We prove that $\Delta(C) \ge 0$. If $\{f(x)|g(x) = f(x), x \in V(G)\}$ consists of even numbers, then $1 \equiv e(T,V(C)) + \sum\limits_{x\in V(C)} f(x) \equiv e(T,V(C)) \pmod 2$, in particular, $e(T,V(C)) \ge 1$. Similarly, if $\{f(x)|g(x) = f(x), x \in V(G)\}$

consists of odd numbers, then we have $1 \equiv e(T, V(C)) + |V(C)| \pmod 2$. Suppose $\{d_G(x)|g(x) = f(x).x \in V(G)\}$ consists of even numbers. Then

$$\begin{aligned} 0 \equiv \sum_{x \in V(C)} d_G(x) &= 2|E(C)| + e(V(C), S \cup T) \\ &\equiv e(S \cup T, V(C)) \pmod 2 \end{aligned} \tag{2.4}$$

Thus $e(S \cup T, V(C)) \equiv 0 \pmod 2$. If $\{d_G(x)|g(x) = f(x), x \in V(G)\}$ consists of odd numbers, then we have $|V(C)| = e(S \cup T, V(C)) \pmod 2$. We consider three cases.

Case1. $e(T, V(C)) \geq 1$ and $e(S, V(C)) \geq 1$. It follows immediately from $0 \leq \theta \leq 1$ that $\Delta(C) \geq 0$.

Case2. $e(T, V(C)) = 0$. We first note that $e(S, V(C)) = e(S \cup T, V(C)) \geq n$ since G is n-edge-connected. By the fact mentioned above, $\{f(x)|g(x) = f(x), x \in V(G)\}$ is not a set of even numbers, and so neither (3b) nor (3d) occurs. If G satisfies (3a), then $\Delta(C) \geq \theta n - 1 \geq 0$ as $e(S, V(C)) \geq n$. Suppose G satisfies (3c). Then we have $e(S, V(C)) \geq n + 1$. Hence $\Delta(C) \geq \theta(n + 1) - 1 \geq 0$. We finally assume that G satisfies (3e). Then it follows from the fact mentioned above that $1 \equiv e(S, V(C)) \pmod 2$. If n is odd, then $m = n$ and so $\Delta(C) \geq \theta n - 1 \geq \theta m - 1 \geq 0$. If n is even, then $e(S, V(C)) \geq n+1$ and $m = n+1$. Hence $\Delta(C) \geq \theta(n+1)-1 = \theta m-1 \geq 0$.

Case3. $e(S, V(C)) = 0$. Note that $e(T, V(C)) = e(S \cup T, V(C)) \geq n$. If G satisfies (3a), then $\Delta(C) \geq (1 - \theta)n - 1 \geq 0$. If $\{d_G(x)|g(x) = f(x), x \in V(G)\}$ consists of even numbers, then $e(T, V(C)) \equiv 1 \pmod 2$. Hence (3b) does not occur. If (3c) holds, then $e(T, V(C)) \geq n + 1$ and so $\Delta(C) \geq (1-\theta)(n+1)-1 \geq 0$. Suppose G satisfies (3d). It is easy to show that we may assume n is even. Since $e(T, V(C)) \equiv 1 \pmod 2$, we have $e(T, V(C)) \geq n+1$, and thus $\Delta(C) \geq (1 - \theta)(n + 1) - 1 = (1 - \theta)m - 1 \geq 0$. Finally we suppose that G satisfies (3e). Then $1 \equiv e(T, V(C)) + |V(C)| \pmod 2$ and $|V(C)| \equiv e(T, V(C)) \pmod 2$, a contradiction. Therefore, (3e) does not occur.

Let $S = T = \emptyset$ and assume $\delta(\emptyset, \emptyset) < 0$. Then $h(\emptyset, \emptyset) > 0$. Since G is connected, it follows from Lemma 2.2 that $g(x) = f(x)$ for all $x \in V(G)$ and $\sum f(x) \equiv 1 \pmod 2$, which contradicts (2). Therefore $\delta(\emptyset, \emptyset) = 0$. Consequently, the proof of the theorem is complete.

We now give some results on factors which can be obtained by Theorem 2.1. □

Proposition 2.3 ((1): Petersen [13] ($r = 3$) and Baebler [3] ($r \geq 4$); and (2): Little, Grant and Holton [11]). *Let G be an $(r-1)$-edge-connected r-regular graph. Then*

(1) *if G has an even number of vertices, then G has a 1-factor; and*

(2) *if G has an odd number of vertices, then $G - v$ has a 1-factor of any vertex v of G.*

Proof. We prove only (2) since (1) can be proved similarly. Put $\theta = 1/r$, and define two functions g and f on $V(G)$ as

$$g(x) = \begin{cases} 0 & x = v \\ 1 & \text{otherwise,} \end{cases} \quad \text{and } f(x) = 1 \text{ for all } x \in V(G),$$

where v is a given vertex of G. Then θ, g, f and $n = r - 1$ satisfy(1a), (2) and (3c) or (3e) of Theorem 2.1 according to the parity of r. Hence G has a (g, f)-factor F. We can easily see that $d_F(v) = 0$. Therefore (2) follows. □

Proposition 2.4 ((1),(2): Gallai [7]; and (3): Bollobás, Saito and Wormald [5]) . *The following statements hold.*

(1) *An n-edge-connected $2r$-regular graph with an even number of vertices has a $(2k+1)$-factor for every $2k+1$,$2r/n \leq 2k+1 \leq 2r(n-1)/n$.*

(2) *An n-edge-connected $(2r+1)$-regular graph G has a $2k$-factor for every $2k$, $0 \leq 2k \leq (2r+1)(n-1)/n$. In particular, G has a $(2m+1)$-factor for every $2m+1$, $(2r+1)/n \leq 2m+1 \leq 2r+1$.*

(3) *A $2n$-edge connected $(2r+1)$-regular graph G has a $2k$-factor for every $2k$, $0 \leq 2k \leq (2r+1)(2n)/(2n+1)$. In particular, G has a $(2m+1)$-factor for every $2m+1$, $(2r+1)/(2n+1) \leq 2m+1 \leq 2r+1$.*

Proof. We prove only (3) since (1) and (2) can be proved similarly. Set $\theta = 2k/(2r+1)$, and define two functions g and f on $V(G)$ by $g(x) = f(x) = 2k$ for all $x \in V(G)$. Then θ, g, f and $2n$ satisfy (1a), (2) and (3d) of Theorem 2.1. Therefore G has a (g, f)-factor, which is a $2k$-factor of G. Let F be a $2k$-factor of G. Then $G - E(F)$ is a $(2r+1-2k)$-factor of G, and so G has a $(2m+1)$-factor for every $2m+1$, $(2r+1)/(2n+1) \leq 2m+1 \leq 2r+1$. Note that the latter can be proved independently by using (3e) of Theorem 2.1. □

3 Proof of Theorem 1.2.

We shall prove Theorem 1.2 by using Theorem 1.1.

Proof of Theorem 1.2. Let G be a $[2a, 2b]$-factorable graph. Then G can be decomposed into $[2a, 2b]$-factors for some positive integer m. It is clear that G is a $[2am, 2bm]$-graph.

Conversely, suppose that G is a $[2am, 2bm]$-graph. We prove that G can be decomposed into $[2a, 2b]$-factors by induction on m. Without loss of generality, we may assume G is connected. Put $\theta = 1/m$, and define two functions g and f on $V(G)$ as follows:

$$g(x) = f(x) = 2a \quad \text{if} \quad d_G(x) = 2am$$

$$g(x) \leq \theta d_G(x) \leq f(x) \text{ with } f(x) - g(x) = 1 \text{ if } 2am < d_G(x) < 2bm, \text{ and}$$

$$g(x) = f(x) = 2b \quad \text{if} \quad d_G(x) = 2bm$$

Then, θ, g, f and $n = 1$ satisfy (1a), (2), and (3b) of Theorem 2.1. Therefore, G has a (g, f)-factor F. For any vertex x of G with $2am < d_G(x) < 2bm$, we have

$$2a < \theta d_G(x) < 2b \quad \textit{and} \quad 2a(m-1) < (1-\theta)d_G(x) < 2b(m-1).$$

Hence F is a $[2a, 2b]$-factor, and $G - E(F)$ is a $[2a(m-1), 2b(m-1)]$-factor. Conversely, the theorem follows by induction.

4 Proof of Theorem 1.3.

In this section we shall prove the following four statements: (i) Every $[8m + 2k, 10m + 2k]$-graph is $[1, 2]$-factorable (Theorem 1.3). (ii) If $r \geq 8m$, then every $[r, r + 2m - 1]$-graph is $[1, 2]$-factorable (Corollary 1.4). (iii) Every connected $[r, r + 1]$-graph is $[1, 2]$-factorable (Corollary 1.4). And (iv) there exists a $[6, 8]$-graph which is not $[1, 2]$-factorable (Remark 4.3).

We first prove Theorem 1.3 under the assumption that the following lemma holds.

Lemma 4.1 *Let G be a $[4, 6]$-graph with at most one vertex of degree 6. Then G can be decomposed into three $[1, 2]$-factors.*

We begin with the next lemma.

Lemma 4.2 *Every $[8, 10]$-graph can be decomposed into six $[1, 2]$-factors.*

Proof. Let G be an $[8, 10]$-graph. Without loss of generality, we may assume G is connected. If G has vertices of degree 10, then choose any vertex w of degree 10. Set $\theta = 1/2$, and define two functions g and f on $V(G)$ by

$$g(x) = \begin{cases} 4 & \text{if } 8 \le d_G(x) \le 9 \\ 5 & \text{otherwise,} \end{cases} \text{ and } f(x) = \begin{cases} 4 & \text{if } d_G(x) = 8 \\ 5 & \text{if } 9 \le d_G(x) \le 10 \text{ and } x \ne w \\ 6 & \text{if } x = w. \end{cases}$$

Then θ, g, f and $n = 1$ satisfy (1a), (2) and (3c) of Theorem 2.1. Hence G has a (g, f)-factor F. It follows that F is a $[4, 6]$-graph with at most one vertex of degree 6 and $G - E(F)$ is a $[4, 5]$-graph, and we conclude by Lemma 4.1 that G can be decomposed into six $[1, 2]$-factors. □

Proof of Theorem 1.3. It follows from Theorem 1.1 and Lemma 4.2 that every $[8m, 10m]$-graph can be decomposed into $6m$ $[1, 2]$-factors. Let G be an $[8m + 2k, 10m + 2k]$-graph with $m \ge 1$ and $k \ge 1$. We may assume G is connected. Put $\theta = 2/(10m + 2k)$ and define two functions g and f on $V(G)$ by

$$g(x) = \begin{cases} 2 & d_G(x) = 10m + 2k \\ 1 & \text{otherwise,} \end{cases} \text{ and } f(x) = 2 \text{ for all } x \in V(G).$$

Then θ, g, f and $n = 1$ satisfy (1a), (2) and (3b) of Theorem 2.1. Hence G has a (g, f)-factor F, which is a $[1, 2]$-factor. Since $G - E(F)$ is an $[8m + 2(k-1), 10m+2(k-1)]$-graph, we conclude by the induction hypothesis that G is decomposed into $6m + k[1, 2]$-factors. □

Proof of Corollary 1.4. We first prove (1). Let H be an $[r, r+2m-1]$-graph with $r \ge 8m$. Then there exist integers k and t such that $r = 8m+2k+t$, $0 \le k$ and $0 \le t \le 1$. It is immediate that $8m+2k \le r$ and $r+2m-1 \le 10m+2k$. Hence F is an $[8m + 2k, 10m + 2k]$-graph, and so H is $[1, 2]$-factorable by Theorem 1.3.

We next prove (2). We first show that every $[2k - 1, 2k]$-graph is $[1, 2]$-factorable. Let G be a $[2k - 1, 2k]$-graph. Then it follows from Theorem 2.1 that G has a $[1, 2]$-factor F such that $d_F(x) = 2$ if $d_G(x) = 2k$ (see Proof of Theorem 1.3). Since $G-E(F)$ is a $[2k-3, 2k-2]$-graph, we have by induction that G is $[1, 2]$-factorable. By the statement (1) and the result given above, it suffices to show that if $r = 2$, 4, or 6, then a connected $[r, r+1]$-graph is $[1, 2]$-factorable. It follows from Lemma 4.5, which will be given later, that every connected $[2, 3]$-graph is $[1, 2]$-factorable. By Lemma 4.1, every $[4, 5]$-graph is $[1, 2]$-factorable. Hence we may restrict ourselves to the case of $r = 6$. Let H be a connected $[6, 7]$-graph. Since a 6-regular graph is 2-factorable, we may assume that H has at least one vertex of degree 7. We show that H can be decomposed into four $[1, 2]$-factors. Put $\theta = 1/2$ and define two functions g and f on $V(H)$ by

$$g(x) = 3 \text{ for all } x \in V(H), \quad \text{and } f(x) = \begin{cases} 3 & \text{if } d_G(x) = 6 \\ 4 & \text{otherwise.} \end{cases}$$

Then θ, g, f and $n = 1$ satisfy (1a), (2) and (3c) of Theorem 2.1. Hence H has a (g, f)-factor F'. It is clear that both F' and $H - E(F')$ are $[3, 4]$-factors of H. Therefore H is $[1, 2]$-factorable.

It is convenient to introduce a new definition. For a set $\{a, b, c, \cdots\}$ of integers, a graph G is called an $\{a, b, c, \cdots\}$*-graph* if $d_G(x) \in \{a, b, c, \cdots\}$ for every $x \in V(G)$. The *union* of graphs H and K is a graph G such that $V(G) = V(H) \cup V(K)$ and $E(G) = E(H) \cup E(K)$.

Remark 4.3 *The following statements hold.*

(1) *A connected* $\{6, 8\}$*-graph having exactly one vertex of degree* 6 *cannot be decomposed into four* $[1, 2]$*-factors.*

(2) *The* 6*-regular graph with three vertices, in which every pair of vertices are joined by exactly three multiple edges, cannot be decomposed into five or more* $[1, 2]$*-factors.*

(3) *The union of a connected* $\{6, 8\}$*-graph with one vertex of degree* 6 *and the* 6*-regular graph given in* (2) *is not* $[1, 2]$*-factorable.*

Proof. We first prove (1). Suppose that connected $\{6, 8\}$-graph G with one vertex v of degree 6 has a $[1, 2]$-factorization $F_1 \cup F_2 \cup F_3 \cup F_4$. Then it follows for some F_i that $d_{F_i}(v) = 1$ and $d_{F_i}(x) = 2$ if $x \neq v$, a contradiction. Statement (2) is immediate. Statement (3) is an easy consequence of (1) and (2). □

In order to prove Lemma 4.1, we shall give some lemmas.

Lemma 4.4 *Every* $[0, 4]$*-graph can be decomposed into two* $[0, 2]$*-factors.*

Proof. Let G be a connected $[0, 4]$-graph. Then G is a $[1, 4]$-graph. We define *theta* $= 1/2$ and two functions g and f on $V(G)$ by

$$g(x) = \begin{cases} 0 & \text{if } d_G(x) = 1 \\ 1 & \text{if } 2 \leq d_G(x) \leq 3 \\ 2 & \text{if } d_G(x) = 4, \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 1 & \text{if } d_G(x) = 1 \\ 2 & \text{otherwise.} \end{cases}$$

Then θ, g, f and $n = 1$ satisfy (1a), (2) and (3b) of Theorem 2.1. Hence G has a (g, f)-factor F, and thus lemma holds since F and $G - E(F)$ are both $[0, 2]$-factors of G.

Lemma 4.5 *Let* G *be a connected* $[2, 4]$*-graph with at least one vertex of degree* 3*. Then* G *can be decomposed into two* $[1, 2]$*-factors.*

Proof. Set $\theta = 1/2$, and define two functions g and f on $V(G)$ by

$$g(x) = \begin{cases} 1 & \text{if } 2 \le d_G(x) \le 3 \\ 2 & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 1 & \text{if } d_G(x) = 2 \\ 2 & \text{otherwise.} \end{cases}$$

Then θ, g, f and $n = 1$ satisfy (1a), (2) and (3b) of Theorem 2.1. Hence G has a (g, f)-factor F, and G is decomposed into two $[1, 2]$-factors F and $G - E(F)$.

The following lemma, which is a special case of Lemma 4.9, shows that Lemma 4.1 holds if the graph is 3-edge-connected. Recall that an $\{a, b, c, \cdots\}$-graph satisfies $d_G(x) \in \{a, b, c, \cdots\}$ for all $x \in V(G)$.

Lemma 4.6 *Let G be a 3-edge-connected $[3, 6]$-graph with at most one vertex of degree 6. Then G has a $[0, 2]$-factorization $F_1 \cup F_2 \cup F_3$ such that if $d_G(x) \ge 4$, then $d_{F_i} \ge 1$ for every F_i.*

Proof. We first assume that G has at least one vertex of degree 3 or 5, or G is a $\{4, 6\}$-graph with an even number of vertices of degree 4. Let $\theta = 1/4$ and define two functions g_1 and f_1 on $V(G)$ by

$$g_1(x) = \begin{cases} 0 & \text{if } d_G(x) = 3 \\ 1 & \text{if } 4 \le d_G(x) \le 5 \\ 2 & \text{otherwise,} \end{cases} \quad \text{and} \quad f_1(x) = \begin{cases} 1 & \text{if } 3 \le d_G(x) \le 4 \\ 2 & \text{otherwise.} \end{cases}$$

Then θ, g_1, f_1 and $n = 3$ satisfy (1b; $\epsilon = 0$, or $1/2$), (2) and (3c) of Theorem 2.1. Hence G has a (g_1, f_1)-factor F_1. It is obvious that $G - E(F_1)$ is a $[2, 4]$-graph with the property that each vertex of degree 2 in $G - E(F_1)$ has degree 3 in G. By Lemma 4.4, $G - E(F_1)$ is decomposed into two $[0, 2]$-factors F_2 and F_3. Consequently, G is decomposed into three $[0, 2]$-factors F_1, F_2 and F_3, which possess the desired property.

We next assume that G is a $\{4, 6\}$-graph with an odd number of vertices of degree 4. It suffices to show that G can be decomposed into three $[1, 2]$-factors. Suppose G is a 4-regular graph. Then it follows from Proposition 2.3 that $G - v$ has a 1-factor L_1 for a vertex v of G. Let F_1 be the $[1, 2]$-factor of G obtained from L_1 by adding an edge of $G - E(L_1)$ incident with v. Since $H_1 = G - E(F_1)$ is a $[2, 3]$-graph having exactly one vertex of degree 2, we have by Lemma 4.5 that H_1 can be decomposed into two $[1, 2]$-factors F_2 and F_3. Therefore, we obtain a required $[1, 2]$-factorization $F_1 \cup F_2 \cup F_3$ of G. Consequently, we may assume that G has exactly one vertex w of degree 6. Set $\theta = 1/4$ and define two functions g_2 and f_2 on $V(G)$ by

$$g_2(x) = f_2(x) = 1 \qquad \text{for all } x \in V(G)$$

Then θ, g_2, f_2 and $n = 3$ satisfy (1b; $\epsilon = 1/2$), (2) and (3c) of Theorem 2.1. Hence G has a (g_2, f_2)-factor L_2. Let F_1 be the $[1, 2]$-factor of G obtained from

L_2 by adding an edge of $G - E(L_2)$ incident with w. Since $H_2 = G - E(F_1)$ is a $[2,4]$-graph having exactly one vertex of degree 4 and one vertex of degree 2, it follows from Lemma 4.5 that H_2 can be decomposed into two $[1,2]$-factors F_2 and F_3. Therefore we obtain a desired $[1,2]$-factorization $F_1 \cup F_2 \cup F_3$ of G. □

We denote by xy and yx an edge joining two vertices x and y. Let G be a graph and v and w be two distinct vertices of G. Then $G + vw$ denotes the graph obtained from G by adding a new edge vw to G, where G may have edges joining v and w. The following Lemmas 4.7 and 4.8 will be used in the proof of Lemma 4.9.

Lemma 4.7 *Let G be a connected $[2,6]$-graph which has exactly one vertex w of degree 2 and at most one vertex of degree 6. Suppose that two distinct vertices u_1 and u_2 are adjacent to w and $G - w + u_1u_2$ is a 3-edge-connected graph. Then G has a $[0,2]$-factorization $F_1 \cup F_2 \cup F_3$ with the property that $d_G(x) \geq 4$ and $d_{F_i}(x) \geq 1$ for every F_i and $d_{F_i}(x) \leq 1$ for every F_i.*

Proof. Let us define two functions g and f on $V(G)$ by

$$g(x) = \begin{cases} 0 & \text{if } 2 \leq d_G(x) \leq 3 \\ 1 & \text{if } 4 \leq d_G(x) \leq 5 \\ 2 & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 1 & \text{if } 2 \leq d_G(x) \leq 4 \\ 2 & \text{otherwise.} \end{cases}$$

We shall show that G has a (g,f)-factor by Lemma 2.2. We denote the vertex of degree 6, if any, by v. Let S, $T \subset V(G)$ such that $S \cap T = \emptyset$ and $S \cup T \neq \emptyset$. We write $\{C_1, \cdots, C_r\}$ for the set of components of $G - (S \cup T)$ which satisfy the conditions on $h(S,T)$ in Lemma 2.2, where $r = h(S,T)$. Then each C_i does not contain w, and so $e(S \cup T, V(C_i)) \geq 3$. Moreover, we have $e(S \cup T, V(C_i)) \geq 4$ since $e(S \cup T, V(C_i)) \equiv 0 \pmod 2$ (see (2.4) in the proof of Theorem 2.1). We obtain the following inequality by setting $\theta = 1/4$ in (2.3) in the proof of Theorem 2.1 (note that (2.3) holds for every graph).

$$\delta(S,T) \geq \sum_{i=1}^{r} \Big\{ \frac{3}{4} e(T, V(C_i)) + \frac{1}{4} e(S, V(C_i)) - 1 \Big\} - \epsilon,$$

where $\epsilon = 0$ or $1/2$ according as $v \notin V(G)$ or $v \in V(G)$. Then

$$\delta(S,T) \geq \sum_{i=1} \Big\{ \frac{1}{4} e(S \cup T, V(C_i)) - 1 \Big\} - \epsilon \geq -\epsilon > -1.$$

Since $\delta(S,T)$ is an integer, we conclude that $\delta(S,T) \geq 0$. It is clear that $\delta(\emptyset,\emptyset) = 0$ as $g(w) < f(w)$. Consequently, G has a (g,f)-factor F. Put $H = G - E(F)$. We consider two cases.

Case1. $d_F(w) = 1$. By Lemma 4.4, H can be decomposed into two $[0,2]$-factors F_2 and F_3, and it is easy to see that $(F_1 = F) \cup F_2 \cup F_3$ is a $[0,2]$-factorization of G with the required property.

Case2. $d_F(w) = 0$. In this case H is a $[2,4]$-graph. Let C be any component of H. If C does not contain w, then we decompose C into two $[0,2]$-factors. Suppose C contains w. Then C contain u_1 and u_2. If $d_C(u_1) = d_C(u_2) = 4$, then we may assume $d_G(u_1) = 5$, and so $F + wu_1$, where $wu_1 \in E(C)$, is also a (g,f)-factor of G. Hence Case1 occurs, and thus we may assume $d_C(u_1) \leq 3$ or $d_C(u_2) \leq 3$. Set $\theta = 1/2$ and define two functions g_1 and f_1 on$V(G)$ by

$$g_1(x) = \begin{cases} 1 & \text{if } d_C(x) \leq 3 \\ 2 & \text{otherwise,} \end{cases} \quad \text{and} \quad f_1(x) = \begin{cases} 1 & \text{if } x = w \\ 2 & \text{otherwise.} \end{cases}$$

Then θ, g_1, f_1 and $n = 1$ satisfy (1a), (2) (since $g_1(u_1) < f_1(u_1)$ or $g_1(u_2) < f_1(u_2)$) and (3c) of Theorem 2.1. Hence C has a (g_1, f_1)-factor, and thus C is decomposed into two $[1,2]$-factors, in each factor of which the degree of w is 1. Therefore G can be decomposed into three $[0,2]$-factors with the required property. □

Lemma 4.8 *Let G be a 3-edge-connected $[3,5]$-graph having a vertex w of degree 3. Then G has a $[0,2]$-factorization $F_1 \cup F_2 \cup F_3$ with the property that if $d_G(x) \geq 4$, then $d_{F_i}(x) \geq 1$ for every F_i and that $d_{F_i}(w) = 0$ for some F_i.*

Proof. Let g and f be functions on $V(G)$ defined by

$$g(x) = \begin{cases} 0 & \text{if } d_C(x) = 3 \\ 1 & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 0 & \text{if } x = w \\ 1 & \text{if } 3 \leq d_G(x) \leq 4 \text{ and } x \neq w \\ 2 & \text{otherwise.} \end{cases}$$

We shall show that G has a (g,f)-factor. Let S, $T \subset V(G)$ such that $S \cap T = \emptyset$ and $S \cup T \neq \emptyset$, and let $\{C_1, \cdots, C_r\}$ be the components of $G - (S \cup T)$ which satisfy the conditions on $h(S,T)$, where $r = h(S,T)$. Then we have the following inequality by setting $\theta = 1/4$ in (2.2).

$$\delta(S,T) \geq (1 - \frac{1}{4}) \sum_{t \in T} d_G(t) + \frac{1}{4} \sum_{s \in S} d_G(s) - \epsilon - e(S,T) - r,$$

where $\epsilon = 0$ or $3/4$ according as $w \notin S$ or $w \in S$. Hence

$$\delta(S,T) \geq \sum_{i=1}^{r} \Big\{ \frac{3}{4} e(T, V(C_i)) + \frac{1}{4} e(S, V(C_i)) - 1 \Big\} - \epsilon,$$
$$\geq \sum_{i} \Big\{ \frac{1}{4} e(T \cup S, V(C_i)) - 1 \Big\} - \epsilon$$

If C_i does not contain w, then $e(T \cup S, V(C_i)) \equiv 0 \pmod 2$ (see 2.4), and so $e(T \cup S, V(C_i)) \geq 4$. Therefore

$$\frac{1}{4} e(T \cup S, V(C_i)) - 1 \geq 0.$$

If C_i contains w, then $\epsilon = 0$ and $e(T \cup S, V(C_i)) \geq 3$, and so

$$\frac{1}{4} e(T \cup S, V(C_i)) - 1 \geq -\frac{1}{4}.$$

Consequently, we obtain $\delta(S,T) \geq -3/4$, which implies $\delta(S,T) \geq 0$. Furthermore, we can show that $\delta(\emptyset, \emptyset) = 0$ by the fact G has at least one vertex x with odd degree except w, for which $g(x) < f(x)$. Consequently, G has a (g,f)-factor F_1. By Lemma 4.4, $G - E(F_1)$ can be decomposed into two $[0,2]$-factors F_2 and F_3. Therefore we obtain a desired $[0,2]$-factorization $F_1 \cup F_2 \cup F_3$ of G. □

We need some notation and definitions in order to prove Lemma 4.1. A graph having exactly two vertices and one or more edges is called a *bond*, and we denote the bond with n edges by B_n (Fig. 4). Let v be a vertex of a graph G and w be a vertex of the bond B_n. Then $G + vw + B_n$ denotes the graph obtained from G and B_n by joining v and w. by a new edge vw (Fig. 4).

We shall prove the next lemma instead of Lemma 4.1, which includes Lemma 4.1 as a special case.

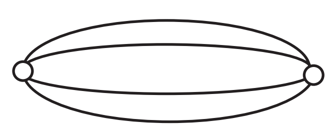

Figure 1: The bond B_4.

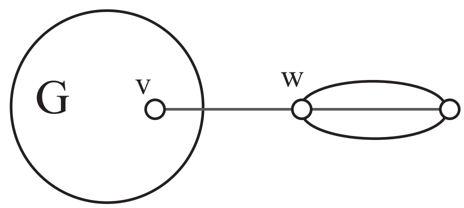

Figure 2: $G + vw + B_3$

Lemma 4.9 *Let G be a connected $[3,6]$-graph with at most one vertex of degree 6. Then G has a $[0,2]$-factorization $F_1 \cup F_2 \cup F_3$ with the property that*

$$\text{if } d_G(x) \geq 4, \text{ then } d_{F_i}(x) \geq 1 \qquad \text{for every } F_i. \tag{4.1}$$

Proof. We prove the lemma by induction on the number of vertices of a graph. Let G be a connected $[3,6]$-graph with at most one vertex of degree 6. By Lemma 4.6, we may assume that G is not 3-edge-connected. If $|V(G)| = 2$ or 3, then G must be 3-edge-connected, which is contrary to the assumption. Hence we may assume $|V(G)| \geq 4$.

First suppose that G is not 2-edge-connected. Then G has a bridge $e = vw$, where $e \in E(G)$ and $v, w \in V(G)$. Let H and K be the components of $G - e$ such that $v \in V(H)$ and $w \in V(K)$. If $|V(H)| \geq 3$ and $|V(K)| \geq 3$, then $H' = H + vu + B_3$ and $K' = K + wu + B_3$ (see Fig. 4) are both $[3,6]$-graphs, where u is a vertex of B_3. By the induction hypothesis, H' and K' can be decomposed into three $[0,2]$-factors with the property (4.1), respectively. It is easy to obtain a desired $[0,2]$-factorization of G from them. Therefore, we may assume $|V(K)| = 2$. Then K is B_3, B_4 or B_5.

If $d_H(v) \geq 3$. then H has a $[0,2]$-factorization with the property (4.1) by induction, and it is easy to obtain a desired $[0,2]$-factorization of G from it. Hence we may assume $d_H(v) = 2$. If two distinct vertices x and y of H are adjacent to v, then $H - v + xy$ (Fig. 4) can be decomposed into three $[0,2]$-factors with the property (4.1) by induction. So we can obtain a desired $[0,2]$-factorization of G from it. We next suppose that one vertex x and v are joined by two edges in H. Let $H + B_3$ be the graph obtained from H by identifying v and one of the vertices of B_3 (Fig. 4). Then, by the induction hypothesis, $H + B_3$ has a $[0,2]$-factorization with the property (4.1), and it is immediate to obtain a desired $[0,2]$-factorization of G from it. Consequently, the proof is complete if G is not 2-edge-connected.

We now deal with the case that G is 2-edge-connected. Since G is not 3-edge-connected, G has a *cutset* (i.e., a minimal cut) with two edges. We consider three cases.

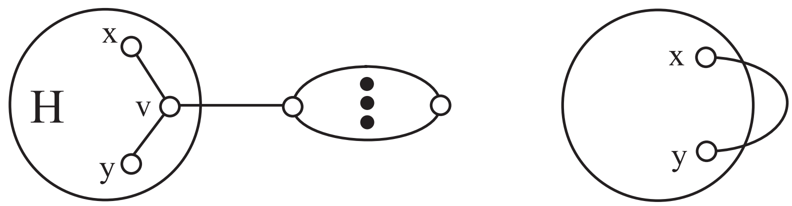

Figure 3: G and $H - v + xy$

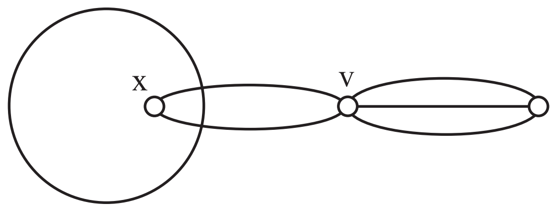

Figure 4: $H + B_4$

Case 1. G has a cutset $\{e_1, e_2\}$ such that the ends of e_1 and those of e_2 are all distinct, where e_1, $e_2 \in E(G)$.

Let H and K be the components of $G - \{e_1, e_2\}$, and let $e_1 = u_1w1$ and $e_2 = u_2w_2$, where u_1, $u_2 \in V(H)$, $u_1 \neq u_2$, w_1, $w_2 \in V(K)$ and $w_1 \neq w_2$. Then $H + u_1u_2$ and $K + w_1w_2$ have $[0, 2]$-factorizations with the property (4.1) by induction. It is easy to obtain a desired $[0, 2]$-factorization of G from them.

Case 2. G has a cutset $\{e_1, e_2\}$ such that the ends of e_1 and those of e_2 are the same. Let H be an arbitrary component of $G - \{e_1, e_2\}$, and v be the end of e_1 and e_2 contained in H. We shall show that H has a $[0, 2]$-factorization $F_1 \cup F_2 \cup F_3$ with the property that

$$F_1 + e_1, F_2 + e_2, \quad \text{and } F_3 \text{ are } [0,2]\text{-factors of } H+ < e_1, e_2 > \\ \text{and satisfy the condition (4.1) in } H+ < e_1, e_2 >, \quad (4.2)$$

where $H+ < e_1, e_2 >$ is the subgraph of G obtained from H by adding e_1 and e_2 together with their common end not contained in H. If this statement follows, then we can easily obtain a $[0, 2]$-factorization of G with the property (4.1) from a $[0, 2]$- factorization of G of each component of $G - \{e_1, e_2\}$.

We now prove the statement. If $d_G(v) \geq 5$, then $d_H(v) \geq 3$ and so H has a $[0, 2]$-factorization $F_1 \cup F_2 \cup F_3$ with the property (4.1) by induction. Since we may assume $d_{F_1}(v) \leq 1$ and $d_{F_2}(v) \leq 1$, these factors satisfy the required condition (4.2). If $d_G(v) = 3$, then G has a bridge, and so Case 1 occurs. Hence we may assume $d_G(v) = 4$, and thus $d_H(v) = 2$. If two distinct vertices x and y of H are adjacent to v, then $H - v + xy$ can be decomposed into three $[0,2]$-factors with the property (4.1) by induction. It is easy to obtain a desired $[0,2]$-factorization of H from them. We next assume that one vertex x of H and v are joined by two edges. Let $H + B_3$ be the graph obtained from H and B_3 by identifying the property (4.1) by induction, and so we can obtain a desired $[0,2]$-factorization of H from it. Consequently, each component of $G - \{e_1, e_2\}$ has a $[0,2]$-factorization satisfying the condition (4.2), and we conclude that the proof of Case 2 is complete.

Case3. For every cutset $\{e_1, e_2\}$ of G, e_1, and e_2 have exactly one vw_2, where v, w_1, $w_2 \in V(G)$ and $w_1 \neq w_2$. Let H and K be the components of $G - \{e_1, e_2\}$ such that $v \in V(H)$ and w_1, $w_2 \in V(K)$. Note that $d_H(v) \geq 2$ as G has no bridges. We first prove that if $\{e_1, e_2\}$ satisfies one of the following two conditions, then G has a $[0,2]$-factorization with the property (4.1):

(i) $K + w_1w_2$ is 3-edge-connected (Fig. 4);

(ii) $d_H(v) = 3$ and H is a 3-edge-connected graph without vertices of degree 6.

Suppose (i) holds. Then $K + < e_1, e_2 >$ (Fig. 4) can be decomposed into three $[0,2]$-factors which satisfy the conditions in Lemma 4.7. On the other hand, if $d_H(v) \geq 3$, then H has a $[0,2]$-factorization with the property (4.1) by induction, and so we can get a desired $[0,2]$-factorization of G. If $d_H(v) = 2$, then two distinct vertices x and y of H are adjacent to v, and so $H - v + xy$ has a $[0,2]$-factorization with the property (4.1). It is easy to obtain a desired $[0,2]$-factorization of G.

We next suppose that (ii) holds. Then H can be decomposed into three $[0,2]$-factors which satisfy the conditions in Lemma 4.8. It follows that $K + w_1w_2$ has a desired $[0,2]$-factorization with the property (4.1) by induction, and thus G has a desired $[0,2]$-factorization.

We shall shoe that G has a cutset $\{e_1, e_2\}$ which satisfies one of the above conditions (i) and (ii). We can choose a cutset $\{e_1, e_2\}$ so that H or $K + w_1w_2$ is 3-edge-connected. If $d_H(v) = 2$, then we may assume without loss of generality that $K + w_1w_2$ is 3-edge-connected. Hence (i) follows.

Suppose $d_H(v) = 3$. In this case we may assume that $K + w_1w_2$ is not 3-edge-connected and H contains a unique vertex of G with degree 6. (otherwise, $\{e_1, e_2\}$ satisfies (i) or (ii)). Let $\{f_1, f_2\}$ be any cutset of $K + w_1w_2$. If the ends of f_1 and those of f_2 are all distinct, then G has such a acutest, which contradicts the assumption of Case 3. If the ends of f_1 and those of f_2 are the same, then f_1 or f_2, say, f_1, must be w_1w_2. So it follows that both $\{e_1, f_2\}$ and $\{e_2, f_2\}$ are cutsets of G and f_2 joins w_1 and w_2. Let T be the component of $G - \{e_1, f_2\}$ containing w_1. If $d_T(w_1) \geq 3$, then T has a $[0, 2]$-factorization with the property (4.1) by induction. If $d_T(w_1) =$

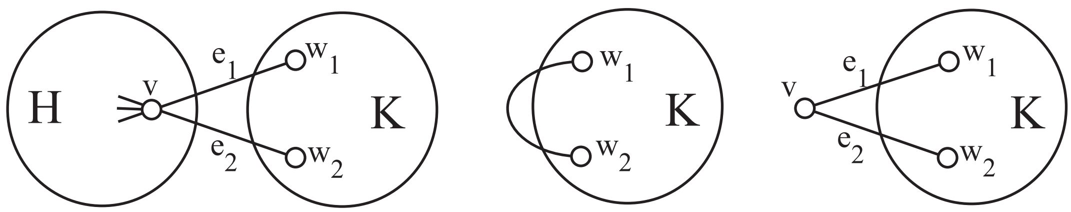

Figure 5: G, $K + w_1w_2$ and $K+ < e_1, e_2 >$

2, then two distinct vertices t_1 and t_2 are adjacent to w_1 in T, and thus $T - w_1 + t_1t_2$ has a $[0, 2]$-factorization with the property (4.1) by induction. Obviously, the component of $G - \{e_2, f_2\}$ containing w_2 has the same property mentioned above. Furthermore, H also has a $[0, 2]$-factorization with the property (4.1). Therefore, we can obtain a desired $[0, 2]$-factorization of G from them. Consequently, we may assume that for every acutest $\{f_1, f_2\}$ of $K + w_1w_2$, f_1 and f_2 have exactly one common end. Hence we can write $f_1 = xy_1$ and $f_2 = xy_2$, where x, y_1, $y_2 \in V(K + w_1w_2)$.

Choose a acutest $\{f_1 = xy_1, f_2 = xy_2\}$ of $K + w_1w_2$ so that the component of $K + w_1w_2 - \{f_1, f_2\}$ containing x is 3-edge-connected or the graph y_2 by adding a new edge y_1y_2 is 3-edge-connected. Since H contains the vertex of degree 6, we can choose such a acutest $\{f_1, f_2\}$ so that the 3-edge-connected component (or graph) has no vertices of degree 6. If $w_1w_2 \notin \{f_1, f_2\}$, then $\{f_1, f_2\}$ is a acutest G which satisfies one of (i) and (ii). Hence we may assume $f_1 = w_1w_2$ and $f_2 = w_1y_2$, where $w_2 \neq y_2$. Then $\{e_2, y_2\}$ is a cutset of G satisfies the condition of Case 1, a contradiction.

We finally assume that $d_H(v) = 4$ (i.e., v is the vertex of G with degree 6.). If $K + w_1w_2$ is 3-edge-connected, then we can obtain a $[0, 2]$-factorization of G with the property (4.1) by applying Lemma 4.8 to $K+ < e_1, e_2 >$. Hence we may assume that $K + w_1w_2$ is not 3-edge-connected. In this case we can prove that G has a desired $[0, 2]$-factorization by the same argument in the case of $d_H(v) = 3$. Consequently, Case 3 is proved.

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References

[1] J.Akiyama, Factorization and linear arboricity of graphs, Doctoral thesis, *Science University of Tokyo* (1982).

[2] J. Akiyama, and M.Kano, Factors and factorizations of graphs-A survey, *J. Graph Theory* **9** (1985) pp.1–42 (this issue)

[3] F.Baebler, Uber die Zerlegung Regul... aren Strechemkoplexe ungerader Ordnung, *Comment Math. Helvetici* **10** (1938) pp. 275-287.

[4] M.Behzad, G. Chartrand and L. Lesniak-Foster, Graphs and Digraphs, Prindle, Weber & Schmidt, Boston (1979).

[5] B.Bollobás, A.Saito and N.C.Wormald, Regular factors of regular graphs. *J. Graph Theory* **9** (1985), pp. 97–103.

[6] H. Era, Semi-regular factorization of regular graphs. Graphs and Applications. In *Proceeding of the First Colorado Symposium of Graph Theory*, F. Harary and J. Maybee, eds., Wiley, New York, in press.

[7] T.Gallai, On factorization of graphs. *Acta Math. Acad. Sci. Hungary.*, **1** (1950), pp. 133–153.

[8] M.Kano and A.Saito, $[a, b]$-factors of graphs. *Discrete Math.*, **47** (1983), pp. 113–116.

[9] M Kano, Graph factors with given properties. Graph Singapore 1983. Lecture Notes in Mathematics 1073. Springer NewYork (1984).

[10] M.Las.Vergnas, An extension of Tutte's 1-factor theorem. *Discrete Math.*, **23** (1978) pp. 241–255.

[11] C.H.Little, D.D.Grant, and D.A.Holton, On defeat d-matchings in graphs. *Discrete Math.* **23** (1975) pp. 41–54; Erratum. *Discrete Math.* **14** (1976) 203.

[12] L.Lovász, Subgraphs with proscribed valances. *J.Combinatorial Theory* **8** (1970) pp. 391–416.

[13] J.Petersen Die Theorie der Regulären Graphs. *Acta.Math* **15** (1891) pp. 193–220.

[14] J.Plesnik, Connectivity of regular graphs and the existence of 1-factors. *Matematikývcasopis* **22** (1972) pp. 310–318.

[15] A.Saito and M.Kano, Some results on factors. Technical Report 82-10, Department of Information Science, Tokyo University (1982).

[16] W.T.Tutte, Graph factors. *Combinatorica* **1** (1981) pp. 79–97.