Ranking the Vertices of a Paired Comparison Digraph *

Mikio Kano[1] and Akio Sakamoto[2]

[1] Department of Mathematics

Akashi Technology College, Uozumi, Akashi674, Japan

[2] Faculty of Engineering

Tokushima University, Minami-josanjima

Abstract

A paired comparison digraph $D = (V, A)$ is a weighted digraph in which the sum of the weights of arcs, if any, joining two distinct vertices is exactly one; there exist no arcs joining them. A one-to-one mapping α from V onto $\{1, 2, \cdots, |V|\}$ is called a ranking of D. We define the backward arcs and the backward length of α. An optimal ranking of D is a ranking whose backward length is minimum among those of all rankings of D. Our method of ranking the vertices of D is one that makes use of these optimal rankings. For certain classes of paired comparison digraphs, we show that the optimal rankings can be explicitly computed.

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1 Introduction

Consider a weighted digraph in which every arc vw has a weight $\epsilon(vw)$. We shall be concerned with a *a paired comparison digraph* (PCD) which is defined to be a weighted digraph satisfying the following conditions (see Fig. 1):

(i) $0 < \epsilon(v, w) \leq 1$ for every arc (v, w) of D.

(ii) $\epsilon(v, w) + \epsilon(w, v) = 1$ if both (v, w) and (w, v) are arcs of D.

(iii) $\epsilon(v, w) = 1$ if (v, w) is an arc of D but (w, v) is not.

A digraph D can be considered as a PCD if we set the weight of every arc of D as follows:

(i) $\epsilon(v, w) = 0.5$ if both vw and wv are the arcs;

(ii) $\epsilon(v, w) = 1$ if vw is an arc but wv is not.

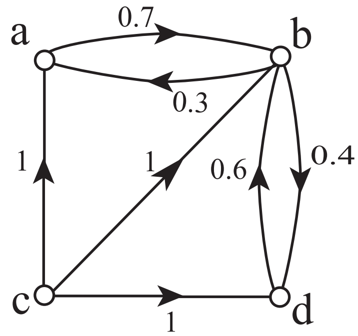

Figure 1: A paired comparison digraph.

Throughout this paper, we regard a digraph as a PCD in this way. In particular, a *tournament*, in which every two vertices are joined by exactly one arc, is a PCD.

A digraph D is a natural way of representing the results of paired experiments; this is, if v is superior to w (v defeats w), then vw us an arc of D but wv is not; if v is equivalent to w (game ends in a draw), then both vw and wv are arcs of D; and if v and w are not compared (game is not played), then there are no arcs joining v and w. Furthermore, we may interpret the weight $\epsilon(vw)$ of an arc vw in a PCD as the rate with which certain consumers prefer v to w in a paired comparison test (with which v defeats w).

We now explain the method of ranking the vertices which will be discussed in this paper. Let D be a PCD with n vertices. A *ranking* α of D is a one-to-one mapping from the set of vertices of D onto the set of integers $\{1, 2, \cdots, n\}$. For ranking α, the image $\alpha(v)$ of a vertex v is called the *rank* of v defined by α, and an arc vw such that $\alpha(w) < \alpha(v)$ is called *backward arc* of α. We write $B(\alpha)$ for the set of backward arcs of α, and define the *backward length* of α, denoted by $\|B(\alpha)\|$, as follows:

$$\|B(\alpha)\| = \sum_{(v,w)\in B(\alpha)} \epsilon(vw)\{\alpha(v) - \alpha(w)\}.$$

Then the backward length of ranking α can be considered as the value of unreasonableness of α. On the other hand, an arc xy with $\alpha(x) < \alpha(y)$ and $\epsilon(xy) = 1$ represents only that a player x with higher rank defeats a player y with lower rank.

For example, let α be a ranking of a PCD in Fig 1 with $\alpha(a) = 1$, $\alpha(b) = 3$, $\alpha(c) = 4$, $\alpha(d) = 2$, then it follows that $B(\alpha) = \{da, ba, cd, cb\}$ and $\|B(\alpha)\| = 1 \times 1 + 0.3 \times 2 + 1 \times 2 + 0.6 \times 1 = 4.2$ (see Fig 1).

A ranking α of D is said to be *optimal* if backward of α is minimum among those of all rankings of D. For a vertex v of D, the average of the ranks defined by the optimal rankings of D is called the *proper rank* of v. Our method of ranking the vertices of D is one that

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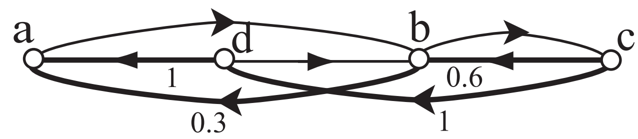

Figure 2: A ranking α

makes use of these proper ranks which depend on the optimal rankings of D. Therefore our ranking procedure can be applied to every PCD, and we shall show that this method has reasonable properties.

There are several methods of ranking the vertices of a tournament. Our approach is to compare the scores, which are the numbers of games won by each player, and compare them. In [8], this ranking method is called the "point system" and characterized by a set of axioms. Another ranking is obtained by making use of the maximum positive eigenvalue and its positive eigenvector, due to Perron and Frobenius, of the adjacency matrix of a tournament (e.g., Wei [9] and Kendall [6], [2, p185], Moon and Pullman [7] Berge [1, p74]). It follows in the former that the ranks of players whose scores are the same are not distinguished. On the other hand, the latter may discriminate the ranks of players having the same score are ignored. We show, however, an example of a tournament in which the vertex, whose rank is determined to be the first by the latter method, does not have a maximum score. It is given in Fig 1. The maximum positive eigenvalue and its positive eigenvector of the adjacency matrix of this tournament are approximately 3.174 and [.165, .157, .180, .130, .119, .083, .083, .083, .083] respectively. Thus the ranking of the vertices is in the order of c, a, b, d, e, $\cdots$. However the scores of a and b are both five and that of c is four.

Note that the ranking methods mentioned above are only for a tournament. On the other hand, our aim in this paper is to consider a ranking procedure applicable to every PCD, including tournaments as a special case.

For a ranking α of a PCD D, we can define the forward arcs of α and its forward length similarly. A ranking α of D is defined to be forward optimal if its forward length is *maximum* among those of all rankings. The forward optimal rankings may be applied to rank vertices of D. But if we regard D as the results of games, then this ranking method seems not as natural as the backward case discussed in the present paper. However, the ranking method with forward length has different properties and may be useful for another application [4].

2 Notation and preliminary results

For finite sets X and Y, we denote the number of elements in X by $|X|$ or $\#\{x \in X\}$, and denote $X \cup Y$ by $X + Y$ if it is a disjoint union. A digraph is said to be *asymmetric* if every two vertices are joined by at most one arc, and is said to be *complete* if every two vertices

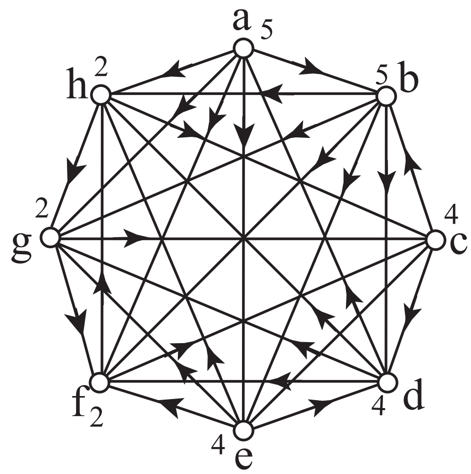

Figure 3: A tournament. Numbers denote the scores

are joined by at least one arc. A complete asymmetric digraph is called a *tournament*. We say that a digraph is *acyclic* if it contains no oriented cycles.

Let $D = (V, A)$ be a PCD with n vertices. We define five functions; $\bar{\epsilon} : V \times V \to [0, 1]$, $\mu : V \times V \to \{0, 1\}$, σ^+ and $\sigma^- : V \to [0, n-1]$, and $d^* : V \to \{0, 1, 2, \cdots, n-1\}$ as follows:

$$\bar{\epsilon}(vw) = \begin{cases} \epsilon(vw) > 0 & vw \text{ is an arc } D, \\ 0 & \text{otherwise,} \end{cases}$$
$$\mu(vw) = \mu(wv) = \bar{\epsilon}(vw) + \bar{\epsilon}(wv),$$
$$\sigma^+(v) = \sum_{x \in V} \bar{\epsilon}(vx) \quad \text{and} \quad \sigma^-(v) = \sum_{x \in V} \bar{\epsilon}(xv),$$
$$d^*(v) = n - 1 - (\sigma^+(v) + \sigma^-(v)) = n - 1 - \sum_{x \in V} \mu(vx).$$

It is obvious that if v and w are compared, then $\mu(vw) = 1$; otherwise, $\mu(vw) = 0$. $d^*(v)$ is the number of vertices which are not compared with v. Since we may regard $\sigma^+(v)$ as a generalized score of v, we call $\sigma^+(v)$ the *positive score* (or briefly *score*) of v and $\sigma^-(v)$ the *negative score* of v. Note that if D is an asymmetric digraph, then the positive and negative scores of v are the out-degree and in-degree of v, respectively.

We put the symbol of the digraph, say D, as a subscript of an appropriate function if necessary. For example, we write $\sigma_D^+(v)$.

If α is a ranking of D with $\alpha(v_i) = i$ for $1 \leq i \leq n$, then we write $\alpha = [v_1, v_2, \cdots, v_n]$. The vertex whose rank defined by a ranking α is k is denoted by $\alpha^{-1}(k)$. We denote the set of optimal rankings of D by $\mathrm{OR}(D)$. The backward length of an optimal ranking of D is called the *optimal backward length* of D and denoted by $l(D)$. Then we have

$$l(D) = \min_{\alpha}\{\|B(\alpha)\|\} \text{ and } \mathrm{OR}(D) = \{\alpha|\ \|B(\alpha)\| = l(D)\}.$$

Note that if H is a subdigraph of D with $V(H) = V(D)$, then $l(H) \le l(D)$ since $B_H(\alpha) \subseteq B_D(\alpha)$ for every α.

Remark 2.1. *A* PCD D *is acyclic if and only if* $l(D) = 0$.

Proof. The remark follows from the fact that an acyclic digraph contains at least one vertex of in-degree zero and that subdigraph of an acyclic digraph is also acyclic.

Remark 2.2. *The optimal backward length of a* PCD D *is the sum of those of the strongly connected components of* D.

Proof. This remark can be proved similarly to Remark 2.1 by regarding each strongly connected component of D as a vertex of an acyclic digraph.

Let α be a ranking of a PCD D and let m and k be integers such that $1 \le k < k+m \le n$, where n is the number of vertices of D. Then we define a ranking α_m^k by

$$\alpha_m^k(v) = \begin{cases} k+m & \text{if } v = \alpha^{-1}(k), \\ k & \text{if } v = \alpha^{-1}(k+m), \\ \alpha(v) & \text{otherwise.} \end{cases}$$

Lemma 2.3. *Let* α *be a ranking of a* PCD *with* n *vertices. If* $\alpha(v) = k$ *and* $\alpha(w) = k+m$, *then*

$$\begin{aligned} \|B(\alpha_m^k)\| - \|B(\alpha)\| &= m\Big(\sigma^+(v) - \sigma^+(w) + \sum_{\alpha(x)>k+m}\{\mu(wz) - \mu(vz)\}\Big) \\ &\quad + \sum_{k<\alpha(y)<k+m}\{\alpha(y) - k\}\{\mu(wy) - \mu(vy)\} \\ &= m\Big(\sigma^-(w) - \sigma^-(v) + \sum_{\alpha(x)<k}\{\mu(vx) - \mu(wx)\}\Big) \\ &\quad + \sum_{k<\alpha(y)<k+m}\{(k+m) - \alpha(y)\}\{\mu(vy) - \mu(wy)\}. \end{aligned}$$

Proof. By the definition, the backward length of α is expressed as

$$\|B(\alpha)\| = \sum_{\alpha(x)<\alpha(y)} \bar{\epsilon}(yx)\{\alpha(y) - \alpha(x)\}$$

where the summation is over all pairs of vertices x and y satisfying $\alpha(x) < \alpha(y)$. Let $X = \{x \in V | \alpha(x) < k\}$, $Y = \{y \in V | k < \alpha(y) < k+m\}$, $Z = \{z \in V | k+m < \alpha(z)\}$, and $B = \{st \in B(\alpha) | \{s,t\} \cap \{v,w\} = \emptyset\}$. Then we have

$$\begin{aligned}
\|B(\alpha)\| = &\sum_{x \in X}[\bar{\epsilon}(vx)\{k - \alpha(x)\} + \bar{\epsilon}(wx)\{(k+m) - \alpha(x)\}] \\
&+ \sum_{y \in Y}[\bar{\epsilon}(yv)\{\alpha(y) - k\} + \bar{\epsilon}(wy)\{(k+m) - \alpha(y)\}] \\
&+ \sum_{z \in Z}[\bar{\epsilon}(zv)\{\alpha(z) - k\} + \bar{\epsilon}(zw)\{\alpha(z) - (k+m)\}] \\
&+ m\bar{\epsilon}(wv) + \sum_{st \in B} \bar{\epsilon}(st)\{\alpha(s) - \alpha(t)\}.
\end{aligned}$$

Moreover, $\|B(\alpha_m^k)\|$ is obtained from the above equation only by interchanging v and w. Hence we have

$$\begin{aligned}
\|B(\alpha_m^k)\| - \|B(\alpha)\| = m\Big(&\sum_{x \in X}\{\bar{\epsilon}(vx) - \bar{\epsilon}(wx)\} + \sum_{y \in Y}\{\bar{\epsilon}(vy) - \bar{\epsilon}(wy)\} \\
&+ \sum_{z \in Z}\{\bar{\epsilon}(zw) - \bar{\epsilon}(zv)\} + \bar{\epsilon}(vw) - \bar{\epsilon}(wv)\Big) \\
&+ \sum_{y \in Y}\{\alpha(y) - k\}\{\mu(wy) - \mu(vy)\}.
\end{aligned}$$

On the other hand, it follows that

$$\begin{aligned}
\sigma^+(v) - \sigma^+(w) = &\sum_{x \in X}\{\bar{\epsilon}(vx) - \bar{\epsilon}(wx)\} + \sum_{y \in Y}\{\bar{\epsilon}(vy) - \bar{\epsilon}(wy)\} \\
&+ \sum_{z \in Z}\{\bar{\epsilon}(vz) - \bar{\epsilon}(wz)\} + \bar{\epsilon}(vw) - \bar{\epsilon}(wv).
\end{aligned}$$

These equations lead to the first equation of the lemma. The second one is obtained similarly by considering $\sigma^-(w) - \sigma^-(v)$.

3 Complete PCD and semicomplete PCD.

We begin with the following lemma which gives us the backward length of any ranking of a complete PCD.

Lemma 3.1. *Let K be a complete* PCD *with n vertices and let α be any ranking of K. Then*

$$\|B(\alpha)\| = \sum_{v \in V} \sigma^+(v)\alpha(v) - \frac{1}{6}n(n^2-1).$$

Proof. We proof the equation by induction on n. The basis, $n = 1$, is obvious. Suppose that the equation holds for $n = k$, and let $n = k+1$. Let x be the vertex such that $\alpha(x) = n$, and put $W = V(K) \setminus \{x\}$. By the induction hypothesis on $K - x$, we have

$$\|B_K(\alpha)\| = \sum_{v \in W} \{\sigma_K^+(v) - \bar{\epsilon}_K(vx)\}\alpha(v) - \frac{1}{6}k(k^2-1) + \sum_{v \in W} \bar{\epsilon}_K(xv)\{n - \alpha(v)\}.$$

Since $\bar{\epsilon}_K(xv) + \bar{\epsilon}_K(vx) = 1$, the last term can be expanded as follows;

$$\begin{aligned}\sum_{v \in W} \bar{\epsilon}_K(xv)\{n - \alpha(v)\} &= n \sum_{v \in W} \bar{\epsilon}_K(xv) + \sum_{v \in W} \{\bar{\epsilon}_K(vx) - 1\}\alpha(v) \\ &= \sigma_K^+(x)n + \sum_{v \in W} \bar{\epsilon}_K(vx)\alpha(v) - \frac{1}{2}k(k+1)\end{aligned}$$

Thus we obtain

$$\begin{aligned}\|B_K(\alpha)\| &= \sum_{v \in W} \sigma_K^+(v)\alpha(v) + \sigma_K^+(x)n - \frac{1}{6}k(k+1)(k+2) \\ &= \sum_{v \in V(K)} \sigma_K^+(v)\alpha(v) - \frac{1}{6}n(n^2-1)\end{aligned}$$

This completes the proof.

By Lemma 3.1, the optimal rankings, the proper ranks, and the optimal backward length of a complete PCD are easily obtained as the following theorem. Remember that the proper rank of v, denoted by $\pi(v)$, is defined by

$$\pi(v) = \frac{1}{|\mathrm{OR}(D)|} \sum_{\alpha \in \mathrm{OR}(D)} \alpha(v)$$

.

Theorem 3.2. *Let K be a complete* PCD *with n vertices. Then*

(1) *A ranking $\alpha = [v_1, v_2, \cdots, v_n]$ of K is optimal if and only if $\sigma^+(v_1) \geq \sigma^+(v_2) \geq \cdots \geq \sigma^+(v_n)$.*

(2) *$\pi(v) = \xi + \frac{1}{2}(\eta + 1)$ where $\xi = \#\{x \in V | \sigma^+(x) > \sigma^+(v)\}$ and $\eta = \#\{y \in V | \sigma^+(y) = \sigma^+(v)\}$.*

(3) *$l(K) = \sum_{v \in V} \sigma^+(v)\alpha(v) - \frac{1}{6}n(n^2-1)$ where $\alpha \in \mathrm{OR}(K)$.*

Proof. (1) Let $\alpha = [v_1, v_2, \cdots, v_n]$ be any optimal ranking of K. If $\sigma^+(v_i) < \sigma^+(v_j)$ for some $i < j$, then by Lemma 2.3, we have $\|B(\alpha^i_{j-1})\| - \|B(\alpha)\| = (j-i)[\sigma^+(v_i) - \sigma^+(v_j)] < 0$ as $\mu(xy) = 1$ for all vertices x and y, which is a contradiction. Thus $\sigma^+(v_1) \geq \sigma^+(v_2) \geq \cdots \geq \sigma^+(v_n)$. On the other hand, $\beta = [w_1, w_2, \cdots, w_n]$ be a ranking which satisfies the condition $\sigma^+(w_1) \geq \sigma^+(w_2) \geq \cdots \geq \sigma^+(w_n)$. Then, we have $\|B(\alpha)\| = \|B(\beta)\|$ by Lemma 3.1. Hence $\beta \in \mathrm{OR}(K)$.

(2) Put $\{y \in V | \sigma^+(y) = \sigma^+(v)\} = \{v = y_1, y_2, \cdots, y_\eta\}$ and $|\mathrm{OR}(K)| = r$. Then it follows from (1) that $\{\alpha(y_1), \alpha(y_2), \cdots, \alpha(y_\eta)\} = \{\xi+1, \xi+2, \cdots, \xi+\eta\}$ for all $\alpha \in \mathrm{OR}(K)$ and $\pi(v) = \pi(y_1) = \pi(y_2) = \cdots = \pi(y_\eta)$. Hence we have

$$\begin{aligned}\eta\pi(v) &= \pi(y_1) + \pi(y_2) + \cdots + \pi(y_\eta) \\ &= \frac{1}{r}\sum_{i=1}^{\eta}\sum_{\alpha\in\mathrm{OR}(K)} \alpha(y_i) = \frac{1}{r}\sum_{\alpha}\sum_{i}\alpha(y_i) \\ &= \frac{1}{r}\sum_{\alpha}\sum_{i}(\xi+i) = \frac{1}{r}\sum_{\alpha}\{\xi\eta + \frac{1}{2}\eta(\eta+1)\} = \xi\eta + \frac{1}{2}\eta(\eta+1).\end{aligned}$$

Hence $\pi(v) = \xi + \frac{1}{2}\eta(\eta+1)$.

Statement (3) follows at once from Lemma 3.1.

Consider, for example, a tournament T in Fig 3. By Theorem 3.2, we have the following: $\mathrm{OR}(T) = \{\alpha = [v_1, v_2, \cdots, v_7] | v_1 = a, \{v_2, v_3\} = \{b, c\}, \{v_4, v_5, v_6\} = \{d, e, f\}$ and $v_7 = g\}$, $|\mathrm{OR}(T)| = 12$, $\pi(a) = 1$, $\pi(b) = \pi(c) = 2.5$, $\pi(d) = \pi(e) = \pi(f) = 5$, $\pi(g) = 7$, and $l(T) = 7$.

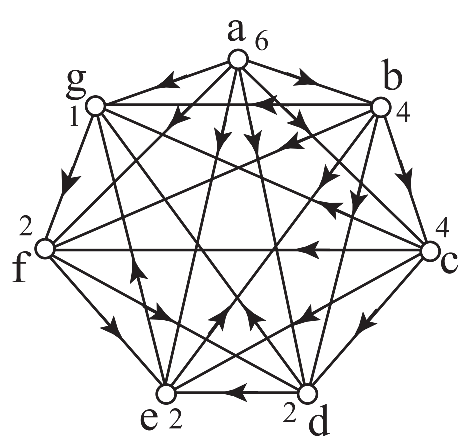

Figure 4: A tournament T. Numbers denote the scores.

If two vertices v and w in a PCD D are not compared (i.e., $\mu(vw) = 0$), then the unordered pair $\{v, w\}$ is called an *uncompared pair* of D. We write $U(D)$ for the set of uncompared pairs of D. If $\{v, w\} \in U(D)$, then D_{vw} denotes the PCD obtained from D by adding a new arc vw of weight one, that is, $D_{vw} = (V(D), A(D) + vw)$. If $\{x, y\}$ is another uncompared pair of D, then $D_{vw,xy}$ is the PCD obtained from D_{vw} by adding a new arc xy of weight one. A complete PCD obtained from D by adding exactly one of arcs vw and wv for each $\{v, w\} \in U(D)$ is called a *completeness* of D. A completeness K of D with $l(K) = l(D)$ is called a *normal completeness* of D, and the set of normal completeness of D is denoted by $\mathrm{NC}(D)$. Note that we can easily see the existence of a normal completeness of D as follows: Let $\alpha \in \mathrm{OR}(D)$. For every $\{v, w\} \in U(D)$, we add a new arc xy to D such that $\{x, y\} = \{v, w\}$ and $\alpha(x) < \alpha(y)$. Then the resulting complete PCD is a normal completeness of D.

A *semicompete* PCD is a PCD in which $d^*(v)$ is either zero or one for every vertex of v. Then a semicomplete digraph is regarded as the result of an incomplete tournament in which each player has not played at most one game. The optimal rankings and the proper ranks pf a semicomplete PCD are obtained by the next theorem. We shall prove it in the next section.

Theorem 3.3. *Let D be a semicomplete* PCD*. Then*

(1) *The set of optimal rankings of D is the disjoint union of the sets of optimal rankings of normal completenesses of D, that is,*

$$\mathrm{OR}(D) = \sum_{K \in \mathrm{NC(D)}} \mathrm{OR}(K). \quad (\textit{disjoint union})$$

.

(2) *A completeness K of D is normal if and only if K satisfies the following conditions:*

(i) *if $\sigma_D^+(v) > \sigma_D^+(w)$ for $\{v, w\} \in U(D)$, then $vw \in A(K)$, and*

(ii) *if $\sigma_D^+(v) = \sigma_D^+(w)$ for $\{v, w\} \in U(D)$, then $A(K)$ contains exactly one of arcs vw and wv.*

In particular, it follows that $|\mathrm{NC}(D)| = 2^r$ where $r = \#\{\{v, w\} \in U(D) | \sigma_D^+(v) = \sigma_D^+(w)\}$.

(3) *For each vertex v of D,*

$$\pi_D(v) = \frac{1}{|\mathrm{NC}(D)|} \sum_{K \in \mathrm{NC}(D)} \pi_K(v)$$

For example, let D be a semicomplete PCD as in Fig 3. Then the normal completeness of D are K_1 and K_2 shown in Fig 3. Hence $\mathrm{OR}(D) = \mathrm{OR}(K_1) + \mathrm{OR}(K_2) = \{\alpha = [x_1, x_2, x_3, c, d] | \{x_1, x_2, x_3\} = \{a, b, e\}\} + \{\beta = [y_1, y_2, y_3, a, d] | \{y_1, y_2, y_3\} = \{b, c, e\}\}$.

4 Optimal rankings

In this section we shall investigate optimal rankings of a PCD and prove Theorem 3.3.

Lemma 4.1. *Let D be a* PCD *and $\{v, w\} \in U(D)$. Then the following statements* (1), (2) *and* (3) *are equivalent:*

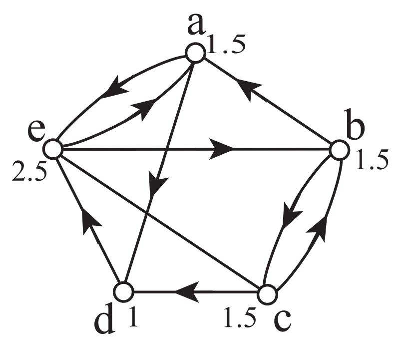

Figure 5: A *semicomplete* PCD D.

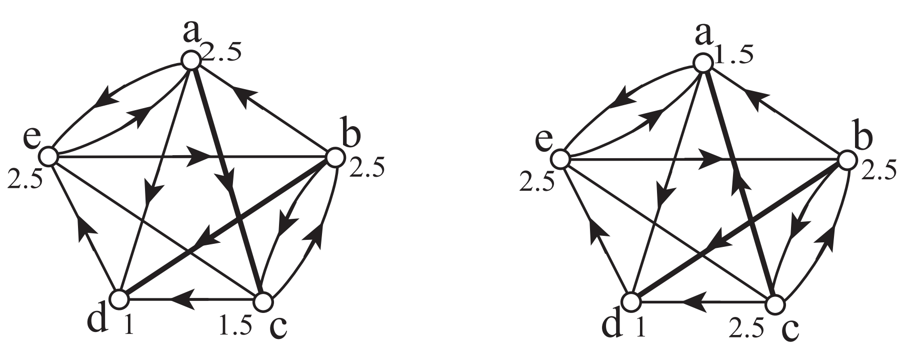

Figure 6: The normal completeness of D.

(1) $l(D_{vw}) < l(D_{wv})$.

(2) $\alpha(v) < \alpha(w)$ *for all* $\alpha \in \mathrm{OR}(D)$.

(3) $\mathrm{OR}(D) = \mathrm{OR}(D_{vw})$ *and* $l(D) = l(D_{vw})$.

Moreover, the next statements (4), (5) *and* (6) *are also equivalent:*

(4) $l(D_{vw}) = l(D_{wv})$.

(5) *There exist two optimal rankings* α *and* β *of* D *such that* $\alpha(v) < \alpha(w)$ *and* $\beta(w) < \beta(v)$.

(6) $\mathrm{OR}(D) = \mathrm{OR}(D_{vw}) + \mathrm{OR}(D_{wv})$ *and* $l(D) = l(D_{vw}) = l(D_{wv})$.

Proof. (1) implies (2): Suppose $l(D_{vw}) < l(D_{wv})$. Assume that there exists $\alpha \in \mathrm{OR}(D)$ such that $\alpha(w) < \alpha(v)$. Then we have

$$l(D) = \|B_D(\alpha)\| = \|B_{D_{wv}}(\alpha)\| \geq l(D_{wv}) > l(D_{vw}).$$

Since D is a subdigraph of D_{vw}, it is clear that $l(D) \leq l(D_{vw})$. This contradicts the above inequality. Thus $\alpha(v) < \alpha(w)$ for all $\alpha \in \mathrm{OR}(D)$.

(2) implies (3): Let $\alpha \in \mathrm{OR}(D)$. Since $\alpha(v) < \alpha(w)$, we have

$$l(D_{vw}) \geq l(D) = \|B_D(\alpha)\| = \|B_{D_{vw}}(\alpha)\| \geq l(D_{vw}),$$

which implies that $l(D) = l(D_{vw})$ and $\alpha \in \mathrm{OR}(D_{vw})$, in particular, $\mathrm{OR}(D) \subseteq \mathrm{OR}(D_{vw})$. Conversely, we suppose $\beta \in \mathrm{OR}(D_{vw})$. Then

$$l(D) = l(D_{vw}) = \|B_{D_{vw}}(\beta)\| \geq \|B_D(\beta)\| \geq l(D).$$

Therefore $\beta \in \mathrm{OR}(D)$, and thus $\mathrm{OR}(D) \supseteq \mathrm{OR}(D_{vw})$.

We prove later that (3) implies (1).

(4) implies (5): Let $l(D_{vw}) = l(D)$. Without loss of generality, we may assume that there is $\alpha' \in \mathrm{OR}(D)$ such that $\alpha'(v) < \alpha'(w)$. We shall derive a contradiction assuming that $\alpha(v) < \alpha(w)$ for all $\alpha \in \mathrm{OR}(D)$. Then we may assume $l(D) = l(D_{vw}) = l(D_{wv})$ since (2) implies (3). Let $\gamma \in \mathrm{OR}(D_{wv})$. If $\gamma(v) < \gamma(w)$, then we have

$$l(D) = l(D_{wv}) = \|B_{D_{wv}}(\gamma)\| = \|B_D(\gamma)\| + \gamma(w) - \gamma(v) > \|B_D(\gamma)\| \geq l(D).$$

This is a contradiction, and thus $\gamma(w) < \gamma(v)$. Hence

$$l(D) = l(D_{wv}) = \|B_{D_{wv}}(\gamma)\| = \|B_D(\gamma)\|,$$

which claims $\gamma \in \mathrm{OR}(D)$. This contradicts the assumption that $\alpha(v) < \alpha(w)$ for all $\alpha \in \mathrm{OR}(D)$.

(5) implies (6): Let $\mathrm{OR}(D)$ be partitioned into two subsets

$$\mathrm{OR}_1 = \{\alpha \in \mathrm{OR}(D) | \alpha(v) < \alpha(w)\} \quad \text{and} \quad \mathrm{OR}_2 = \{\alpha \in \mathrm{OR}(D) | \alpha(w) < \alpha(v)\}$$

.

We shall prove that if there exists $\alpha \in \mathrm{OR}(D)$ satisfying $\alpha(v) < \alpha(w)$, then $\mathrm{OR}(D_{vw}) = \mathrm{OR}_1$ and $l(D) = l(D_{vw})$. Let $\alpha \in \mathrm{OR}_1$. Then we have

$$l(D_{vw}) \geq l(D) = \|B_D(\alpha)\| = \|B_{D_{vw}}(\alpha)\| \geq l(D_{vw}),$$

which implies that $l(D) = l(D_{vw})$ and $\alpha \in \mathrm{OR}(D_{vw})$. In particular $\mathrm{OR}(D_{vw}) \supseteq \mathrm{OR}_1$. On the other hand, let $\beta \in \mathrm{OR}(D_{vw})$. Then,

$$l(D) = l(D_{vw}) = \|B_{D_{vw}}(\beta)\| \geq \|B_D(\beta)\| \geq l(D),$$

which means that $\beta \in \mathrm{OR}(D)$ and $\beta(v) < \beta(w)$. Therefore we have $\beta \in \mathrm{OR}_1$ and thus $\mathrm{OR}(D_{vw}) \subseteq \mathrm{OR}_1$. This completes the proof of $\mathrm{OR}(D_{vw}) = \mathrm{OR}_1$. Therefore (6) holds.

(6) implies (4): This is immediate.

(3) implies (1): Let us assume $l(D_{vw}) \geq l(D_{wv})$. Then we have $l(D_{vw}) = l(D_{wv}) = l(D)$ since $l(D_{wv}) \geq l(D) = l(D_{vw})$. Hence (4) holds and thus (6) follows, which is contrary to (3). Therefore (3) implies (1).

Lemma 4.2. *Let D be a* PCD *and $\{v, w\} \in U(D)$. Then we have the following :*

(1) $l(D) = \min\{l(D_{vw}), l(D_{wv})\}$.

(2)

$$\mathrm{OR}(D) = \begin{cases} \mathrm{OR}(D_{vw}) & \text{if } l(D_{vw}) < l(D_{wv}), \\ \mathrm{OR}(D_{wv}) & \text{if } l(D_{vw}) > l(D_{wv}), \\ \mathrm{OR}(D_{vw}) + \mathrm{OR}(D_{wv}) & \text{if } l(D_{vw}) = l(D_{wv}). \end{cases}$$

(3) *If $l(D) = l(D_{vw})$, then $\mathrm{OR}(D) \supseteq \mathrm{OR}(D_{vw})$ and $\alpha(v) < \alpha(w)$ for every $\alpha \in \mathrm{OR}(D_{vw})$.*

Proof. These statements are easy consequences of Lemma 4.1. Note that there exist D and D_{vw} such that $l(D) < l(D_{vw})$ and $\mathrm{OR}(D) \supseteq \mathrm{OR}(D_{vw})$. Therefore, $l(D) < l(D_{vw})$ implies neither $\mathrm{OR}(D) \not\supseteq \mathrm{OR}(D_{vw})$ nor $\mathrm{OR}(D) \cap \mathrm{OR}(D_{vw}) = \emptyset$. Furthermore, $\mathrm{OR}(D) \supseteq \mathrm{OR}(D_{vw})$ does not imply $l(D) = l(D_{vw})$. For example, let T be a tournament in Fig 3 and D be the semicomplete PCD which is obtained from T by deleting the arc ad. Then it follows from Theorems 3.2 and 3.3 that $l(D) = l(T) = \tau < l(D_{da}) = 10$ and $\mathrm{OR}(D) \supset \mathrm{OR}(D_{da})$.

Remember that $\mathrm{NC}(D)$ denotes the set of normal completenesses K of D, which satisfy $l(K) = l(D)$.

Theorem 4.3. *Let D be a* PCD*. If $\mathrm{NC}(D) = \{K_1, K_2, \cdots, K_r\}$, then $\mathrm{OR}(D) = \mathrm{OR}(K_1) + \mathrm{OR}(K_2) + \cdots + \mathrm{OR}(K_r)$.*

Proof. Let $\{a, b\}$ and $\{c, d\}$ be uncompared pairs of D. Then it follows from Lemma 4.2 that

$$\mathrm{OR}(D) = \Sigma\mathrm{OR}(D_{vw}) = \Sigma\mathrm{OR}(D_{vw,xy}) \qquad \text{(disjoint union)},$$

where the first summation is over D_{vw} such that $\{v, w\} = \{a, b\}$ and $l(D_{vw}) = l(D)$, and the second summation is over $D_{vw,xy}$ such that $\{v, w\} = \{a, b\}$, $\{x, y\} = \{c, d\}$, and $l(D_{vw,xy}) = l(D)$. By repeating this procedure, we obtain the theorem.

We now give a result which shows a virtue of our method. For example, statement (1) of the following theorem tells us that v is stronger than w if the score v is greater than that of w even though w wins all unplayed $d^*(w)$ games, that is, $\sigma^+(v) > \sigma^+(w) + d^*(w)$.

Theorem 4.4. *Let D be a* PCD *and v and w be two vertices of D. Then:*

(1) *If $\mu(vw) = 1$ and $\sigma^+(v) > \sigma^+(w) + d^*(w)$, then $\alpha(v) < \alpha(w)$ for every $\alpha \in \mathrm{OR}(D)$.*

(2) *If $\mu(vw) = 0$ and $\sigma^+(v) > \sigma^+(w) + d^*(w) - 1$, then $\alpha(v) < \alpha(w)$ for every $\alpha \in \mathrm{OR}(D)$. In particular, $\mathrm{OR}(D) = \mathrm{OR}(D_{vw})$.*

(3) *If $\sigma^+(v) > \sigma^+(w)$ and $\{x \in V | \mu(vx) = 1\} = \{y \in V | \mu(wy) = 1\}$, then $\alpha(v) < \alpha(w)$ for every $\alpha \in \mathrm{OR}(D)$.*

Proof. We first prove (1). By Theorem 3.2, we may assume D is not complete. By Theorem 4.3, every optimal ranking α of D is one of a certain normal completeness K of D. Since $\sigma_K^+(w) \le \sigma_D^+(w) + d_D^*(w) < \sigma_D^+(v) \le \sigma_K^+(v)$, it follows from Theorem 3.2 that $\alpha(v) < \alpha(w)$. Hence (1) follows.

We next prove (2). It follows from Lemma 4.2 that every optimal ranking α of D is one of a certain PCD H which is obtained from D by adding exactly one of the arcs xy and yx for each $\{x, y\} \in U(D) \setminus \{\{v, w\}\}$, and satisfies $l(H) = l(D)$. Suppose $\alpha(w) = k < \alpha(v) = k + m$. Since $\mu_H(vx) = \mu_H(wx) = 1$ for every vertex $x (\neq v, w)$, we have by Lemma 2.3 that $\|B_H(\alpha_m^k)\| - \|B_H(\alpha)\| = m\{\sigma_H^+(w) - \sigma_H^+(v)\} < 0$ as $\sigma_H^+(w) \le \sigma_D^+(w) + d_D^*(w) < \sigma_D^+(v) \le \sigma_H^+(v)$. This is contrary to $\alpha \in \mathrm{OR}(H)$. Consequently, $\alpha(v) < \alpha(w)$.

Statement (3) can be proved similarly by Lemma 2.3.

In order to prove Theorem 3.3 we require the following lemma.

Lemma 4.5. *Let v and w be vertices of a* PCD *D. Then:*

(1) *If $\sigma^+(v) > \sigma^+(w)$, $d^*(v) = d^*(w) = 1$, and $\mu(vw) = 0$, then $\alpha(v) < \alpha(w)$ for all $\alpha \in \mathrm{OR}(D)$, In particular, $\mathrm{OR}(D) = \mathrm{OR}(D_{vw})$ and $l(D) = l(D_{vw})$.*

(2) *If $\sigma^+(v) = \sigma^+(w)$, $d^*(v) = d^*(w) = 1$, $\mu(vw) = 0$ and $\alpha(v) = k < \alpha(w) = k + m$ for $\alpha \in \mathrm{OR}(D)$, then $\alpha_m^k \in \mathrm{OR}(D)$. In particular, $\mathrm{OR}(D) = \mathrm{OR}(D_{vw}) + \mathrm{OR}(D_{wv})$ and $l(D) = l(D_{vw}) = l(D_{wv})$.*

Proof. Statement (1) is a corollary of (3) in Theorem 4.4. By Lemma 2.3 we have $\|B(\alpha)\| - \|B(\alpha_m^k)\| = 0$, and thus (2) follows from Lemma 4.1.

Proof of Theorem 3.3. Statement (1) follows at once from Theorem 4.3. Statement (2) is an easy consequence of Lemma 4.5. We now prove (3). Let K and K' be any two normal completenesses of D. If $\{v, w\} \in U(D)$, then we have $\{\sigma_K^+(v), \sigma_K^+(w)\} = \{\sigma_{K'}^+(v), \sigma_{K'}^+(w)\}$ by (2). Hence $\#\{v \in V(K) | \sigma_K^+(v) = t\} = \#\{v \in V(K') | \sigma_{K'}^+(v) = t\}$ for every positive real number t, and thus it follows from Theorem 3.2 that $|\mathrm{OR}(K)| = |\mathrm{OR}(K')|$. By Theorem 4.3

we have

$$\begin{aligned}\pi_D(v) &= \frac{1}{|\mathrm{OR}(D)|}\sum_{\alpha\in\mathrm{OR}(D)}\alpha(v)\\ &= \frac{1}{|\mathrm{NC}(D)|\cdot|\mathrm{OR}(K)|}\sum_{K\in\mathrm{NC}(D)}\sum_{\alpha\in\mathrm{OR}(K)}\alpha(v)\\ &= \frac{1}{|\mathrm{NC}(D)|}\sum_{K}\Big\{\frac{1}{|\mathrm{OR(K)}|}\sum_{\alpha}\alpha(v)\Big\} = \frac{1}{|\mathrm{NC}(D)|}\sum_{K}\pi_K(v).\end{aligned}$$

5 2-semicomplete asymmetric digraphs

A 2-*semicomplete* PCD is a PCD in which $d^*(v)$ is less than or equal to two for every vertex v. Then a 2-semicomplete asymmetric digraph may represent the result of an incomplete tournament in which no game ended in a draw and each player has not played at most two games. We show some theorems by which we can obtain the optimal rankings of a 2-semicomplete asymmetric digraph. We omit, however, their proofs because they are rather complicated and long. Slightly generalized results of these theorems together with their complete proofs will found in [5].

For a PCD D, we write $(V(D), U(D))$ for the graph whose vertex set is $V(D)$ and whose edge set is the set $U(D)$ of uncompared pairs of D. It is clear that if D is a 2-semicomplete PCD, then each component of D of $(V(D), U(D))$ with more than one vertex is either a path or a cycle. A sequence $[v_1, v_2, \cdots, v_r]$ of vertices of a PCD D is called an *uncompared path* of D if $[v_1, v_2, \cdots, v_r]$ is the sequence of vertices of a path from v_1 to v_r in $(V(D), U(D))$. Similarly, an *umcompared cycle* $[v_1, v_2, \cdots, v_r]$ with r vertices of D can be defined.

Let D be a 2-semicomplete asymmetric digraph and $[v, w]$ be an umcompared path of D such that $d^*(v) = 1$ and $d^*(w) = 1$ or 2. Then we have by Theorem 4.4 that if $\sigma^+(v) > \sigma^+(w) + 1$, then $\mathrm{OR}(D) = \mathrm{OR}(D_{vw})$; and if $\sigma^+(v) < \sigma^+(w)$, then $\mathrm{OR}(D) = \mathrm{OR}(D_{wv})$. Hence we may restrict ourselves to the case that $\sigma^+(v) = \sigma^+(w)$ or $\sigma^+(v) = \sigma^+(w) + 1$.

Theorem 5.1. *Let D be a 2-semicomplete asymmetric digraph and $[v, w = w_1, w_2, \cdots, w_r, u]$ $(r \geq 1)$ be an uncompared path with $d^*(v) = 1$. Suppose $\sigma^+(w) = \sigma^+(w_2) = \cdots = \sigma^+(w_r) = t$ for some integer t. Then the following statements hold:*

(1) *If $\sigma^+(v) = t + 1$ and $\sigma^+(u) \geq t + 1$, then*

$$\mathrm{OR}(D) = \begin{cases}\mathrm{OR}(D_{vw}) + \mathrm{OR}(D_{wv}) & \text{if } r \text{ is even,}\\ \mathrm{OR}(D_{vw}) & \text{otherwise.}\end{cases}$$

(2) *If $\sigma^+(v) = t + 1$ and $\sigma^+(u) \leq t - 1$ or if $\sigma^+(v) = t + 1$ and $\sigma^+(u) = t$ and $d^*(u) = 1$, then*

$$\mathrm{OR}(D) = \begin{cases} \mathrm{OR}(D_{vw}) & \textit{if } r \textit{ is even,} \\ \mathrm{OR}(D_{vw}) + \mathrm{OR}(D_{wv}) & \textit{otherwise.} \end{cases}$$

(3) *If* $\sigma^+(v) = t$ *and* $\sigma^+(u) \geq t+1$, *then*

$$\mathrm{OR}(D) = \begin{cases} \mathrm{OR}(D_{wv}) & \textit{if } r \textit{ is even,} \\ \mathrm{OR}(D_{wv}) + \mathrm{OR}(D_{vw}) & \textit{otherwise.} \end{cases}$$

(4) *If* $\sigma^+(v) = t$ *and* $\sigma^+(u) \leq t-1$ *or if* $\sigma^+(v) = \sigma^+(u) = t$ *and* $d^*(u) = 1$, *then*

$$\mathrm{OR}(D) = \begin{cases} \mathrm{OR}(D_{wv}) + \mathrm{OR}(D_{vw}) & \textit{if } r \textit{ is even,} \\ \mathrm{OR}(D_{wv}) & \textit{otherwise.} \end{cases}$$

Theorem 5.2. *Let D be a 2-semicomplete asymmetric digraph and $[v, w, u]$ be an uncompared path with $d^*(v) = d^*(u) = 2$ or an uncompared of D. Then*

(1) *if* $\sigma^+(v) \geq \sigma^+(w) + 1 \geq \sigma^+(u) + 1$, *then* $\mathrm{OR}(D) = \mathrm{OR}(D_{vw}) = \mathrm{OR}(D_{wv,wu})$ *and*

(2) *if* $\sigma^+(v) \geq \sigma^+(w) + 1 \leq \sigma^+(u)$, *then* $\mathrm{OR}(D) = \mathrm{OR}(D_{vw}) = \mathrm{OR}(D_{vw,uw})$

Theorem 5.3. *let D be a 2-semicomplete asymmetric digraph and $[v = v_1, w = v_2, v_3, \cdots, v_r]$ $(r \geq 3)$ be an uncompared cycle of D. If every vertex v_i has the same score, then* $\mathrm{OR}(D) = \mathrm{OR}(D_{vw}) + \mathrm{OR}(D_{wv})$.

Let D be a 2-semicomplete asymmetric digraph and C be any uncompared cycle of D. Then, we may assume that the length of C is greater than three by Theorems 5.2 and 5.3, and so we can take an uncompared path $[x, v_1, v_2, v_3, \cdots, v_r, y]$ $(r \geq 2)$ such that $\sigma^+(x) < \sigma^+(v_1) = \sigma^+(v_2) = \cdots = \sigma^+(v_r) > \sigma^+(y)$ by the above two theorems. In this case, we can use the following theorem or Theorem 4.4.

Theorem 5.4. *Let D be a 2-semicomplete asymmetric digraph and $[x, v = v_1, w = v_2, v_3, \cdots, v_r, y]$ $(r \geq 2)$ be an uncompared path of D. Suppose $\sigma^+(x) = t-1$, $\sigma^+(v_1) = \sigma^+(v_2) = \cdots = \sigma^+(v_r) = t$ and $\sigma^+(y) = t-1$ for some integer t. Then*

$$\mathrm{OR}(D) = \begin{cases} \mathrm{OR}(D_{vw}) + \mathrm{OR}(D_{wv}) & \textit{if } r \textit{ is even,} \\ \mathrm{OR}(D_{vw}) & \textit{otherwise.} \end{cases}$$

6 Ranking-equal PCD and NP-hardness

A PCD is said to be *balanced* if $\sigma^+(v) = \sigma^-(v)$ for every vertex v, and is said to be *ranking equal* if the proper ranks of all vertices are the same, that is, $\pi(v) = (|V| + 1)/2$ for every vertex v. For a ranking $\bar{\alpha}$ such that $\bar{\alpha}(v) = n + 1 - \alpha(v)$ for each vertex v.

Theorem 6.1. *Let D be a* PCD *with at least one arc. Then every ranking of D is optimal if and only if D is balanced and complete.*

Proof. If D is a balanced complete PCD, then every ranking of D is optimal by Theorem 3.2. Conversely, we suppose that every ranking of D is optimal. Put $n = |V|$. For every two distinct vertices v and w, let us consider a ranking α such that $\alpha(v) = 1$ and $\alpha(w) = 2$. By Lemma 2.3 we have $\|B(\alpha_1^1)\| - \|B(\alpha)\| = \sigma^-(w) - \sigma^-(v)$. Since α and α_1^1 are both optimal, we have $\sigma^-(v) = \sigma^-(w)$. Similarly, considering a ranking β such that $\beta(v) = n-1$ and $\beta(w) = n$, we obtain $\sigma^+(w) = \sigma^+(v)$. Hence there exist two constants δ and δ' such that $\sigma^+(v) = \delta$ and $\sigma^-(v) = \delta'$ for every vertex v. Since $\Sigma\sigma^+(v) = \Sigma\sigma^-(v)$, we have $\delta = \delta'$ and conclude that D is balanced. We next prove D is complete. We may clearly assume $n \geq 3$. For every three distinct vertices u,v and w, let us consider a ranking of α such that $\alpha(u) = 1$, $\alpha(v) = 2$ and $\alpha(w) = 3$. By Lemma 2.3 $\|B(\alpha_1^2)\| - \|B(\alpha)\| = \sigma^-(w) - \sigma^-(v) + \mu(vu) - \mu(wu)$. Since $\|B(\alpha_1^2)\| = \|B(\alpha)\|$ and $\sigma^-(w) = \sigma^-(v)$, we have $\mu(vu) = \mu(wu)$. Therefore $\mu(xy)$ is constant (i.e., $\mu(xy) = 1$ or 0) for every two vertices x and y. Since D has at least one arc, we conclude that D is complete.

Lemma 6.2. *Let α be a ranking of a* PCD *$D = (V, A)$. Then*

$$\|B(\alpha)\| - \|B(\bar{\alpha})\| = \sum_{vw \in A} \epsilon(vw)\{\alpha(v) - \alpha(w)\} = \sum_{v \in V} \alpha(v)\{\sigma^+(v) - \sigma^-(v)\}$$

Proof. Since $B(\bar{\alpha}) = A \setminus B(\alpha)$ and $\bar{\alpha}(v) - \bar{\alpha}(w) = \alpha(w) - \alpha(v)$, we have

$$\begin{aligned}
\|B(\alpha)\| - \|B(\bar{\alpha})\| &= \sum_{vw \in B(\alpha)} \epsilon(vw)\{\alpha(v) - \alpha(w)\} - \sum_{vw \in B(\bar{\alpha})} \epsilon(vw)\{\bar{\alpha}(v) - \bar{\alpha}(w)\} \\
&= \sum_{vw \in A} \epsilon(vw)\{\alpha(v) - \alpha(w)\} = \sum_{v,w \in V} \bar{\epsilon}(vw)\{\alpha(v) - \alpha(w)\} \\
&= \sum_{v \in V}\{\sum_{w \in V} \bar{\epsilon}(vw)\alpha(v)\} - \sum_{w \in V}\{\sum_{v \in V} \bar{\epsilon}(vw)\alpha(w)\} \\
&= \sum_{v} \sigma^+(v)\alpha(v) - \sum_{w} \sigma^-(w)\alpha(w) = \sum_{v} \alpha(v)\{\sigma^+(v) - \sigma^-(v)\}.
\end{aligned}$$

Theorem 6.3. *A* PCD *D is ranking equal if and only if the reversed ranking of every optimal ranking of D is also optimal.*

Proof. We first suppose $D = (V, A)$ is ranking equal. Since $\pi(v) = \pi(w)$ for every two vertices v and w, it follows that

$$\sum_{\alpha \in \mathrm{OR}(D)} \alpha(v) = \sum_{\alpha \in \mathrm{OR}(D)} \alpha(w).$$

Hence we have the following equation by Lemma 6.2

$$
\begin{aligned}
0 &= \sum_{vw \in A} \epsilon(vw) \Big[\sum_{\alpha \in \mathrm{OR}(D)} \{\alpha(v) - \alpha(w)\} \Big] \\
&= \sum_{\alpha} \Big[\sum_{vw} \epsilon(vw) \{\alpha(v) - \alpha(w)\} \Big] = \sum_{\alpha} \{\|B(\alpha)\| - \|B(\bar{\alpha})\|\}.
\end{aligned}
$$

Since $\|B(\alpha)\| \leq \|B(\bar{\alpha})\|$ for every $\alpha \in \mathrm{OR}(D)$, we obtain $\|B(\alpha)\| = \|B(\bar{\alpha})\|$. Hence $\bar{\alpha}$ is also optimal.

We next suppose that $\bar{\alpha}$ is optimal for every $\alpha \in \mathrm{OR}(D)$. Then we have the following equation for any vertex v of D.

$$
\pi(v) = \frac{1}{|\mathrm{OR}(D)|} \sum_{\alpha \in \mathrm{OR}(D)} \alpha(v) = \frac{1}{|\mathrm{OR}(D)|} \sum_{\alpha \in \mathrm{OR}(D)} \bar{\alpha}(v).
$$

Hence we have

$$
2\pi(v) = \frac{1}{|\mathrm{OR}(D)|} \sum_{\alpha} \{\alpha(v) + \bar{\alpha}(v)\} = \frac{1}{|\mathrm{OR}(D)|} \sum_{\alpha} (n+1) = n+1,
$$

where n is the number of vertices of D. Therefore $\pi(v) = (n+1)/2$ and thus D is ranking equal.

Corollary 6.4. *If D is a balanced* PCD*, then D is ranking equal.*

Proof. By Lemma 6.2, $\bar{\alpha} \in \mathrm{OR}(D)$ for all $\alpha \in \mathrm{OR}(D)$. Hence the corollary follows from the above theorem.

The converse statement of this corollary is not true. A counterexample of the converse statement is obtained easily in the following way: Let D be a balanced PCD and vw be an arc of D. We often obtain a nonbalanced ranking-equal PCD D' form D by changing the weights of vw (and wv if it exists) as follows; $\epsilon_{D'}(vw) = \bar{\epsilon}_D(vw) + \delta$ and $\epsilon_{D'}(wv) = \bar{\epsilon}_D(wv) - \delta$ where δ is a sufficiently small positive real number. Furthermore, there exists a counterexample even if we restrict a PCD to a digraph. It is shown in Fig 6. It is verified to be ranking equal by using a computer. We have, however, no counterexample in the case of an asymmetric digraph.

Theorem 6.5. *A problem of finding the optimal backward length of a* PCD *is* NP*-hard.*

Proof. The following problem, called "simple optimal linear arrangement", is NP-complete [3].

> **Input**: Graph $G = (V(G), E(G))$ and positive integer k.
> **Property**: There is a ranking α such that
>
> $$\sum_{v,w \in E(G)} |\alpha(v) - \alpha(w)| \leq k.$$

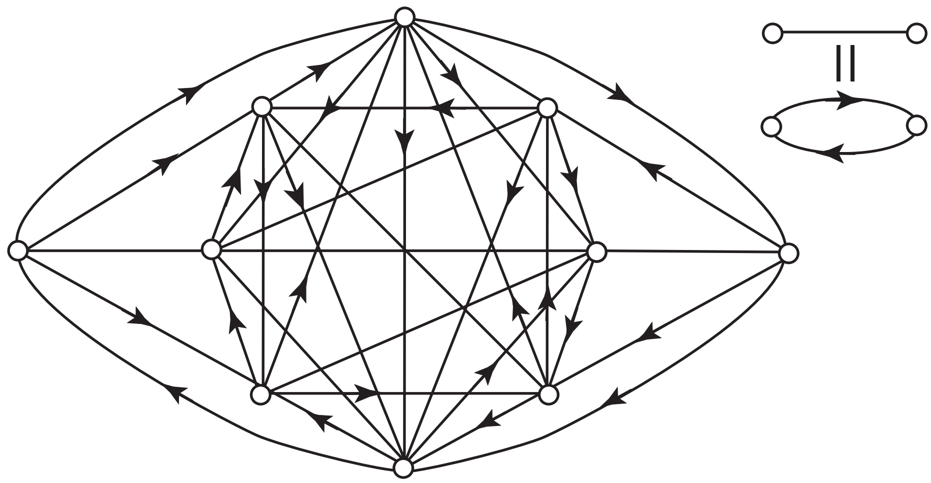

Figure 7: An example of a nonbalanced ranking-equal digraph.

For a graph G given as the input for "simple optimal linear arrangement", let $D = (V(D), E(D))$, where $V(D) = V(G)$. $A(D) = \{vw, wv | \{v, w\} \in E(G)\}$, and the weight of each arc is 0.5. Then we have

$$\sum_{\{v,w\} \in E(G)} |\alpha(v) - \alpha(w)| = 2 \sum_{vw \in B_D(\alpha)} 0.5|\alpha(v) - \alpha(w)| = 2\|B_D(\alpha)\|.$$

Therefore, there exists a ranking α which satisfies the property of "simple optimal linear arrangement" with integer k if and only if there exists a ranking α such that $\|B_D(\alpha)\| \leq k/2$. That is, "simple optimal linear arrangement" is polynomial reducible to the problem of finding the optimal backward length. Hence the theorem is proved.

7 Concluding remarks

We have proposed a new method of ranking the vertices of a paired comparison digraph. This method can be applied not only to a tournament but also to every digraph. Some good properties of the method are given in Remark 2.1, Theorems 3.2 and 4.4 and Corollary 6.4. The basic idea of proof technique is indicated in Lemmas 2.3 and 4.2. However, a problem of finding the optimal backward length of a PCD is NP-hard as seen in Theorem 6.5.

It seems to be dangerous to decide the ranks of participants taken in a round-robin tournament if a lot of games are unplayed because of accidents or the like. We can fortunately obtain the optimal rankings of a PCD D by Lemma 4.2 and Theorems 3.2 and 4.3 if the number of uncompared pairs of D, which corresponds to the number of unplayed games, is small. The optimal rankings of the example given in Fig 6 are obtained by a computer in this way. We have shown in Theorem 3.3 that if $d^*(v)$ which corresponds to the number of games which have not been played by v, is less than or equal to one for every vertex v of a PCD D, then it is easy to obtain the optimal rankings of D. Furthermore, if D is asymmetric digraph in which $d^*(v)$ is less than or equal to two, then we can get the optimal rankings of D by theorems in §5.

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