CYCLE GAMES AND CYCLE CUT GAMES

M. Kano
Department of Mathematics,
Akashi Technologiacl College
Uozumi, Akashi 674, Japan

Abstract

Two players play a game on a connected graph G. Each player in his turn occupies an edge of G. The player who occupies a set of edges that contains a cycle, before the order does iy, wins. This game may end in a draw. We call this game the normal cycle cut game and the misère cycle cut game. We characterize the above four games.

1 Introduction

Let G be a connected graph which may have mutiple edges but has no loops. In this player a graph means such a graph. Two players play a game on G. They take turns and each player in his turn occupies an edge of G not occupied before. We define the following four games:

Normal cycle game: A player wins if he occupies a set of edges that contains a cycle, before the other does it. If the set of edges occupied by each player by the end of the game contains no cycles, then the game ends in a draw.

Misère cycle game: A player loses if he occupies a set of edges that contains a cycle, before the other does it. This game may also end in a draw. The following two games are played between the *cycle player* and the *sut player* or between the *non-cycle player* and the *anti-cut player*.

Normal cycle cut game: The cycle player wins if he occupies a set of edges that contains a cycle. The cut player wins otherwise, that is, the cut player wins if the set of edges not occupied by him contains no cycles. This game never ends in a draw.

Misère cycle cut game: The non-cycle player loses if he occupies a set of edges that contains a cycle. The anti-cut player loses otherwise. Then the goal of the non-cycle player is to avoid a set of edges containinig a cycle and that of the anti-cut player is to force the non-cycle player to occupy a set of edges containing a cycle. This game also never ends in a draw. In this paper, we shall give complete graph theoretic solutions to the above four games, except of the constructive winning strategy for the normal cycle game on certain graphs. We now describe the relation between the cycle cut game and the Shannon switching game ([2],[3],[4],[7]).

The *Shannnon switching game* can be regarded as a normal cycle cut game in which the cycle player, called the *short player*, wins if he occupies a set of X of edges such that the union of X and the distinguished edge e (or e is one of the distinguished edges which form a complete graph $K_n, n \geq 3$) has a cycle containing e. The cut player wins otherwise. Related games are shown in [5]

2 Preliminaries

Let X and Y be subsets of a finite set Z. Then we write $|X|$ for the number of elements in X and denote $X \cup Y$ by $X + Y$ if it is a disjoint union. The complement of X is denoted by $\bar{X}$, that is, $\bar{X} = Z \setminus X$. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$, which may have multiple edges but has no loops. In this paper, for convenience, we call a cycle a *tieset* and consider a tieset, a cutset and a spanning tree of G as subsets of $E(G)$. Let P be a spanning tree of G and let x and y be edges of G s uch that $x \in P$and $y \notin P$. Then $P + y$ has a unique tieset $T(P + y)$, which contains y. Similarly, $\bar{P} + x$ contains a unique cutset $C(\bar{P} + x)$ of G, which contains x, where $\bar{P} = E(G) \setminus P$. THe distance between two spanning trees of G is maximum, then they are said to be *maximally distant*. The set of unordered pairs of maximally distant spanning trees of G is denoted by $\Omega(G)$. Then $(A, B) \in \Omega(G)$ implies that A and B are spanning trees of G abd that $|A \setminus B| \geq |P \setminus Q|$ for every two spanning trees P and Q of G.

Lemma 1 *Let G be a connected graph and $(A, B) \in \Omega(G)$. Let $e \in A \cup B$ and G' be the graph obtained from G by contracting e. Then $(A-e, B-e) \in \Omega(G')$. Furthermore G' has no loops.*

Proof. This lemma follows from the fact that if P' is a spanning tree of G', then $P' + e$ is a spanning tree of G.

The following lemma, which is in fact a theorem on the principal partition of a graph [6], is important.

Lemma 2 (Baron and Imrich [1], Kishi and Y.Kajitani [6]) *A pair of maximally disjoint spanning trees of a connected graph is obtained by a constructive procedure.*

Lemma 3 *Let G be a connected graph and $(A, B) \in \Omega(G)$. Let P be a spanning tree of G and X be a subset of $E(G) \setminus P$. If $|X| \geq |V(G)| - |A \cap B|$, then X contains a tieset.*

Proof. Suppose X contains no tiesets. Then there exists a spanning tree such that $X \subset Q$, and so $|Q \setminus P| \geq |X| \geq |V(G)| - |A \cup B| > |A| - |A \cap B| = |A \setminus B|$, a contradiction. Hence, X contain a tieset.

Lemma 4 *Let G be a connected graoh and $X = \{x_1, cdots, x_r\}$ bea set of edges of G. Let $G_0 = G$ and G_{i+1} $(0 \leq i \leq r-1)$ be the graph obtained from G_i by deleting some edges not in X and by contructing x_{i+1} and some edges not in X. If every x_j $(j > i)$ is not a loop of G_i, for all i, then X contains no tieset of G.*

Proof. It is obvious that if X contains a tieset, then some x_j becomes a loop of some G_i $(j > i)$. Hence the lemma follows.

3 CYCLE GAMES

We shall first consider the cycle games. The normal cycle game and the misère cycle game end without draws if one of the players occupies a set of edges that contains a tieset. Then it can be shown easily that a connected graph G is a graph on which no cycle game ends in a draw if and only if $E \neq AcupB$ for $(A, B) \in \Omega(G)$ (i.e. the arboricity of G is at least three). (See Figs 3 and 3.) We call the player going first the *first player* and the player going second the *second player*.

Figure 1: Graphs on which no cycle game ends in a draw.

Figure 2: Graphs on which no cycle games may end in a draw.

Theorem 1 *Suppose two players play the normal cycle game on a connected graph G. Let $(A, B) \in \Omega(G)$. Then the following statements hold.*

(1) If $E(G) = A \cup B$, then the game ends in a draw if both players make no errors, and there exists a constructive strategy by which each player does not lose. (2) If $E(G) \neq A \cup B$, then the first player can win. Furthermore, if $|E(G)| = |A \cup B| + 1$, then there exists a constructive winning strategy for the first player.

Note that the author does not think that there exists a constructive winning strategy for the first player in case of $|E(G)| \geq |A \cup B| + 2$.

Proof of Theorem 1. We first note that the first player does not lose theoretically in such a game. This fact is well-known (see the beginning of [7]). Suppose $E(G) = AsupB$. We verify only that the second player does not lose. It suffices to show that the second player does not lose even if the first player occupies all the edges in $A \cap B$ before the biginning of the game. Let G' be the graph obtained from G by constructing all the edges in $A \cap B$, and let $A' = A \setminus (A \cap B)$ and $B' = B \setminus (A \cap B)$. Then it follows from Lemma1 that $(A', B') \in \Omega(G')$ and $E(G') = A' + B'$. We may assume that the first player occupies $f_1 \in A'$ in his first turn. Then the second player occupies $s_1 \in T(B' + f_1) - f_1$. Let G_1 be the graph obtained from G' by contracting f_1 and by deleting s_1, and let $A_1 = A' - f_1$and $B_1 = B' - s_1$. Then $(A_1, B_1) \in \Omega(G_1)$ and $e(G_1) = A_1 + B_1$. The second player repeats the above strategy on G_i $(i \geq 1)$. Then $E(G_i) = A_i + B_i$ for $(A_i, B_i) \in \Omega(G_i)$, and so G_i has no loops. Hence, by Lemma4, the set of edges which have been occupied by the first player does not contain a tieset of G, and we conclude that the second player does not lose.

We next assume $E(G) \neq A \cup B$. Then the game never ends in a draw, and thus the first player can win. If $E(G) = (A \cup B) + e$, then the first player can win by occupying e in his first turn and by playing on $G - e$ using the same strategy for the second player in (1).

Theorem 2 *Suppose two players play the misère cycle game on a connected graph G. Let $(A, B) \in \Omega(G)$. Then the following statements hold.*

(1) *If $E(G) = A \cup B$, then the game ends in a draw if the players make no errors.*

(2) *If $E(G) \neq A \cup B$ and $|A \cap B|$ is even, in particular, $A \cap B = \emptyset$, then the second player can win.*

(3) *If $E(G) \neq A \cup B$ and $|A \cap B|$ is odd, then the first player can win.*

(4) *In each case there exists a constructive strategy.*

Proof. Suppose $E(G) = A \cup B$. We first prove that the first player does not lose. It is sufficient to show that the first player does not lose even if he occupies all the edges in $A \cap B$ before the beginning of the game. The first player occupies $f_1 \in A \setminus (A \cap B)$ in his first turn. Let G_1 be the graph obtained from G by contructing all the edges in $(A \cap B) + f_1$, and let $e_1 \in T(B + f_1) \setminus \{(A \cap B) + f_1\}$, $A_1 \in A \setminus \{(A \cap B) + f_1\}$ and $B_1 \in B \setminus \{(A \cap B) + e_1\}$. Then $E(G_1) = A_1 + B_1 + e_1$ and $(A_1, B_1) \in \Omega(G_1)$. Suppose the second player occupies s_1 in his turn. If $s_1 = e_1$, then the first player occupies $f_2 \in A_1$, and let $A_2 = A_1 - f_2$, $e_2 \in T(B_1 + f_2) - f_2$ and $B_2 = B_1 - e_2$. If $s_1 \in A_1$, then the first player occupies $f_2 \in C(\bar{A}_1 + s_1) \cap B_1$, and let $A_2 = A_1 - s_1$, $B_2 = B_1 - f_2$ and $e_2 = e$. We obtain the graph G_2 from G_1 by deleting s_1 and by contructing f_2. Then $E(G_2) = A_2 + B_2 + e_2$ and $(A_2, B_2) \in \Omega(G_2)$. Hence we can show that if the first player repeats the above strategy on G_i $(i \geq 2)$ until the end of the game, then the first player does not lose. Similarly, we can prove that the second player does not lose. Consequently, (1) holds.

Next suppose $E(G) \neq A \cup B$ and $|A \cap B|$ is even. Assume that the first player occupies f_1 in his first turn. If $f_1 \in A \cap B$, then the second player occupies $s \in (A \cap B) - f_1$, , and let $A_1 = A - f_1 - s_1$ and $B_1 = B - f_1 - s_1$. If $f_1 \in A \setminus B$, then the second player occupies $s_1 \in C(\bar{A} + f_1) \cap B$, and let $A_1 = A - f_1$ and $B_1 = B - s_1$. If $f_1 \notin A \cap B$, then the second player occupies $s_1 \in A \setminus B$, and let $A_1 = A - s_1$, and $e_1 \in T(B + s_1) \setminus A$ and $B_1 = B - e_1$. We obtain the graph G_1 from G by contracting s_1 and f_1 or by contracting s_1 and deleting f_1, so that $(A_1, B_1) \in \Omega(G_1)$ and $|A_1 \cap B_1|$ is even. The second player repeats the above strategy on G_i $(i \geq 1)$ until this strategy cannot be used, that is, until one of the following conditions occurs for some r:

(i) $A_r = B_r \neq \emptyset$ (i.e. G_r is a tree having some loops) and the first player occupies $f_{r+1} \notin A_r$; or

(ii) $A_r = B_r = \emptyset$ (i.e. G_r is a graph having exactly one vertex and some loops) and the first player occupies f_{r+1}.

We probe in each case that the set $F = \{f_1, \cdots, f_{r+1}\}$ of edges which have been occupied by the first player contains a tieset of G and that the set $S = \{s_1, \cdots, s_r\}$ of edges which have been occupied by the second player does not contain a tieset of G. It follows from the strategy that $S \subset A \cup B$ and $S \cup (A \cap B)$ is a spanning tree of G, which implies that S has no tiesets of G. We divide F and S as shown in Fig. and write a small letter for the number of edges in each subset. If the case (ii) occurs, then $d = 0$. It follows that $x = y$, $|F| = a + b + c + x = |S| + 1$, $|S| = e + y + f$ and $|S \cup (A \cap B)| = x + d + y + e + f = |V(G)| - 1$. Set $F^* = F \setminus (A \cap B)$.Then we have $|F^*| = a + b + c = |V(G)| - |A \cap B|$. Since $F^* \subset E(G) \setminus (S \cup (A \cap B))$ and $S \cup (A \cap B)$ is a spanning tree of G, we consider that F^* has a tieset of G by Lemma 3. Consequently, the second player can win. If $E(G) \neq A \cup B$ and $|A \cap B|$ is odd, then the first player can win by occupying an edge in $A \cap B$ in his first turn.

Figure 3:

Theorem 3 *Suppose that the cycle player and the cut player play the normal cycle cut game on a connected graph G. Let $(A, B) \in \Omega(G)$. Then the following statements hold:*

(1) $E(G) = A \cup B$, *then the cut player can win.*

(2) $|E(G)| = |A \cup B| + 1$, *then the first player can win.*

(3) $|E(G)| \geq |A \cup B| + 2$, *then the cycle player win.*

(4) *In each case there exists a constructive winning strategy.*

Proof. The statements (1) and (2) are easy consequences of Theorem 1 and its proof. It is clear that if $|E(G)| \geq |A \cup B| + 1$ and the cycle player goes first, thenhe can win. Suppose $|E(G)| \geq |A \cup B| + 2$ and the cut pleyer goes first. We assume that the cut has a component H such that $|E(H)| \geq |A' \cup B'| + 1$ for $(A', B') \in \Omega(H)$. Therefore, the cycle player can win, and thus (3) is proved.

Theorem 4 *Suppose that the non-cycle player and the anti-cut player play the misère cycle cut game on a connected graph G. Let $(A, B) \in \Omega(G)$. Then the following hold:*

(1) *$E(G) = A \cup B$, then the non-cycle player can win.*

(2) *$|E(G)| = |A \cup B| + 1$, and $|A \cap B|$ is even, then the second player can win.*

(3) *$|E(G)| = |A \cup B| + 1$, and $|A \cap B|$ is odd, then the first player can win.*

(4) *$|E(G)| \geq |A \cup B| + 2$, then the anti-sut player win.*

(5) *In each case there exists a constructive winning strategy.*

Proof. The statement (1) follows from (1) of Theorem 2 and its proof. Suppose $|E(G)| = |A \cup B| + 1$ and $|A \cap B|$ is even. Then we can show that the second player can win by using the same strategy for the second player in (2) of Theorem 2 (since we have in this case that $E(G) = \{A_r + f_{r+1}\}$ or $E(G) = \{f_{r+1}\}$ in the proof of Theorem 2). If $|E(G)| = |A \cup B| + 1$ and $|A \cap B|$ is odd, then it follows from (3) that the first player can win by occupying an edge in $A \cap B$ in his first turn.

Finally suppose $|E(G)| \geq |A \cup B| + 2$. We first assume that the anti-cut player goes first. $|A \cap B|$ is odd, then the anti-cut player can win by (3) of Theorem 2. If $|A \cap B|$ is even, then the anti-cut player can win by occupying $f_1 \in E(G) \setminus (A \cup B)$ in his first turn since the graph obtained from G by deleting f_1 satisfies the condition in (2) of Theorem 2. We can prove similarly that the anti-cut player can win if he goes second.

Acknowledgement. I would like to think the referees for their suggestions.

References

[1] G.Baron and W.Imrich, On the maximal distance of the spanning trees, *J. Combinatorial Theory* **5** (1968), pp. 378–385.

[2] J.Bruno and L.Weinberg, A constructive graph-theoric solution of the Shannon switching game, *IEEE Trans. Circuit Theory* **17** (1971), pp. 74–81.

[3] J.Edmonds, Lehman's switching game and a theorem of Tutte and Nash-William *J.Res.NBS* **69B** (1965), pp. 73–77.

[4] M.Kano, Generalizations of Shannon switching game, to appear.

[5] M.Kano, A class of Shannnon switching game played on vertices, *Memoirs of The Akashi Technological College* **19** (1977), pp. 77–85.

[6] G.Kishi and Y.Kajitani, Maximally distant trees and principal partition of a linear graph, *IEEE Trans. Circuit Theory* **16** (1969), pp. 323–330.

[7] A.Lehman, A solution to the Shannnon switching game, *SIAM J.Appl.Math.*, **12** (1964), pp. 687–725.