NOTE

[a,b]-FACTORS OF GRAPHS

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Abstract

For integers a and b such that $0 \leq a \leq b$, a graph G is called an $[a,b]-$graph if $a \leq d_G(x) \leq b$ for every vertex x of G and a factor F of a graph is called an $[a,b]-$factor if $a \leq d_F(x) \leq b$ for every vertex x of F. We prove the following theorems. Let $0 \leq l \leq k \leq r$, $0 \leq s$, $0 \leq u$ and $1 \leq t$. Then an $[r, r+s]-$graph has a $[k, k+t]$-factor if $ks \leq rt$. Moreover, if $(k-l)s + (r-k)u \leq (r-1)t$, then an $[r, r+s]-$graph has a $[k, k+t]$-factor which contains a given $[l, l+u]-$factor.

We consider finite graphs which may have loops and multiple edges.Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex x of a subgraph H of G, we write $d_H(x)$ for the degree of x in H. In particular, $d_G(x)$ is the degree of x. Let a and b be integers such that $0 \leq a \leq b$. Then G is called an $[a,b]-$graph if $a \leq d_G(x) \leq b$ for all $x \in G$.Similarly, a spanning subgraph F of a graph G is called an $[a,b]-$factor of G if $a \leq d_F(x) \leq b$ for all $x \in G$. Let g and f be integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in G$. Then a $(g,f)-$factor of G is spanning subgraph F such that $g(x) \leq d_F(x) \leq f(x)$ for all $x \in G$. For a subset S of $V(G)$, we write $G - S$ for the subgraph G obtained from G by deleting the vertices in S together with edges incident to vertices in S. If S and T are disjoint subsets of $V(G)$, then $e_G(S,T)$ denotes the number of edges joining S and T.

Tutte[4] has shown that an $r-$regular graph (i.e. an $[r,r]-$graph) has a $[k, k+1]-$factor for every k satisfying $0 \leq k \leq r$. Recently, Thomassen[3] gave a simple elegant proof to the following theorem, which is an extension of the above theorem: an $[r, r+1]-$graph has a $[k, k+1]-$factor for every k satisfying $0 \leq k \leq r$. In this paper we shall first prove the next result which is an extension of the theorems mentioned above.

Theorem 1. *Let Gbe a graph and θ be a real number such that $0 \leq \theta \leq 1$. Suppose two integer* *valued functions g and f defined on $V(G)$ satissfy*

$$g(x) < f(x) \qquad and \qquad g(x) \leq \theta d_G(x) \leq f(x) \tag{1}$$

for all $x \in V(G)$. Then G has a $(g, f)-$factor.

Note that when $g(x) \leq f(x)$ instead of $g(x) < f(x)$, the sufficient condition, which is similar to Theorem 1, for the existence of a $(g, f)-$factor is rather complicated [2]. If G is an $[r, r+s]-$graph and $ks \leq rt$, then (1) in Theorem 1 holds for $g(x) \equiv k$, $f(x) \equiv k+t$ and $\theta = \frac{k}{r}$, and we obtain the next corollary.

Corollary 1. *Let $0 \leq k \leq r$, $0 \leq s$ and $1 \leq t$, If $ks \leq rt$, then an $[r, r+s]-$graph has a $[k, k+1]-$factor.*

In the proof of Theorem 1, we shall use the following $(g, f)-$factor theorem due to Lovász, to which Tutte [5] gave a short proof by using his $f-$factor theorem.

Lemma (Lovasz [1], [5, Theorem 7.2]). *Let Gbe a graph and g and f be integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. Then G has a $(g, f)-$factor if and only if*

$$\sum_{t \in T} d_G(t) - \sum_{t \in T} g(t) + \sum_{s \in S} f(s) - e_G(S, T) - h(S, T) \geq 0 \tag{2}$$

for all disjoint subset S and T of $V(G)$, where $h(S, T)$is the number of components C of $G - (S \cup T)$ such that $g(c) = f(c)$ for all $c \in V(C)$ and

$$e_G(T, V(C)) + \sum_{c \in V(C)} f(c) \equiv 1 (mod 2).$$

Note that the condition $0 \leq g(x) \leq f(x) \leq d_G(x)$ in [1], [5] can be replaced by $g(x) \leq f(x)$ as in the theorem. Furthermore, if $g(x) < f(x)$ for every vertex x, then $h(S, T) = 0$ and so the condition (2) becomes simple.

Proof of Theorem 1. Let $S, T \subset V(G), S \cap T = \emptyset$. Then $h(S, T) = 0$ and we have

$$\begin{aligned} & \sum_{t \in T} d_G(t) - \sum_{t \in T} g(t) + \sum_{s \in S} f(s) - e_G(S, T) \\ \geq \quad & (1-\theta) \sum_{t \in T} d_G(t) - \theta \sum_{s \in S} f(s) - e_G(S, T \\ \geq \quad & (1-\theta) e_G(T, S) - \theta e_G(S, T) - e_G(S, T) = 0 \end{aligned}$$

Therefore, (2)holds and we conclude that G has a (g, f)-factor.□

We remark that the condition in corollary 1 is best possible.

Remark. The complete bipartite graph $K_{r,r+s}$ does not have a $[k,k+1]$-factor for any k,t such that $0\le k\le r, 1\le t$ and $ks\ge rt$.

Proof. Let G denote the complete bipartite graph $K_{r,r+s}$. Set S and T be the partite sets of G such that $|S|=r$ and $|T|=r+s$, and put $g\equiv k$ and $f\equiv k+t$. Then

$$\sum_{t\in T} d_G(t)-\sum_{t\in T} g(t)+\sum_{s\in S} f(s)-e_G(S,T)-h(S,T)=r(r+s)-k(r+s)+(k+t)r-r(r+s)=rt-ks<0.$$

Then G does not have a $[k,k+t]$-factor. □

We next give a sufficient conddition for a graph to have a factor which contains a given graph.

Theorem 2. *Let* $0\le l\le k\le r$, $l+u\le k+l\le r+s$, $r\ne l, r+s\ne l+u$, $0\le s, 0\le u$ *and* $1\le t$. *Let* G *be an* $[r,r+s]-$*graph and* H *be an* $[l,l+u]-$*factor of* G. *If*

$$(l-k)s+(k-r)u+(r-l)t\ge 0 \tag{3}$$

then G *has a* $[k,k+1]-$*factor which* H *as a graph.*

If $s=u=t$, then (3) follows, and so we obtain the next corollary.

Corollary 2. *Let* $0\le l\le k\le r$ *and* $1\le t$. *Then an* $[r,r+t]-$*graph has a* $[k,k+t]-$*factor which contains a given* $[l,l+t]-$*factor.*

Proof of Theorem 2. Let G be an $[r,r+1]-$graph and H be an $[l,l+t]-$factor of G. Let $G^{‘}=G-E(H)$ and g and f be functions from $V(G^{‘})$ into the set of integers defined by

$$g(x)=k-d_H(x) \quad and \quad f(x)=k+t-d_H(x)$$

for all $x\in V(G^{‘})$. If $G^{‘}$ has a $(g,f)-$factor F, then $F+E(H)$ is a dedired $[k,k+t]-$factor. Put

$$\theta=\frac{k-l}{r-l} \quad and \quad \lambda=\frac{k+l-l-u}{r+s-l-u}$$

We shall show that $G^{‘}$, g, f and θ satisfy (1) in Theorem 1. It is clear that $0\le\theta\le 1$ and $g(x)<f(x)$ for all $x\in V(G^{‘})$. Since

$$k=\theta r+(1-\theta)\le\theta d_G(x)+(1-\theta)d_H(x) \text{ and } d_G(x)=d_{G^{‘}}(x)-d_H(x),$$

we have $g(x)\le\theta d_{G^{‘}}(x)$. Similarly, we have $f(x)\ge\lambda d_{G^{‘}}(x)$ as

$$k+t=\lambda(r+s)+(1-\lambda)(l+u)\ge\lambda d_G(x)+(1-\lambda)d_H(x).$$

By (3), we obtain $\lambda \geq \theta$ and so $f(x) \geq \theta d_{G'}(x)$. Consequently, (1) holds and the theorem follows. □

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