Draft

Structure Theorem and Algorithm on $(1, f)$-odd subgraphs

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Abstract

The authors give a Gallai-Edmonds type structure theorem on $(1, f)$-odd subgraphs and a polynomial algorithm for finding an optimal $(1, f)$-odd subgraph. Lovász [5] and Cornuéjols [2] solved these problems for a more general problem, the general factor problem with gaps at most 1. However, the statements of the theorems and the algorithm are much more simple in this special case, so it is worth of interest on its own. Also, the algorithm given for this case is faster than the general algorithm. The proofs follow a direct approach instead of deducing from the general case.

1 Introduction

We consider finite graphs without multiple edges or loops. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex v of G, let $\deg_G(v)$ denote the degree of v in G. Let f always denote a function $f : V(G) \to \{1, 3, 5, \dots\}$ such that $f(x) \le \deg_G(x)$ for all $x \in V(G)$. A subgraph H of G is called a $(1, f)$*-odd subgraph* if $\deg_H(x) \in \{1, 3, \dots, f(x)\}$ for all $x \in V(H)$. A spanning $(1, f)$-odd subgraph is called a $(1, f)$*-odd factor*. A $(1, f)$-odd subgraph H said to be *maximum* if there exists no $(1, f)$-odd subgraph H' in G such that $|V(H')| > |V(H)|$.

Lovász [5] and Cornuéjols [2] deal with general factors which is a common generalisation of all factor problems. For each vertex $v \in V(G)$, let B_v be a subset of $\{0, 1, 2, \dots, \deg_G(v)\}$. The *general factor problem* asks whether there exists a spanning subgraph F of G such that for each vertex v we have $\deg_F(v) \in B_v$. With this terminology a $(1, f)$-odd factor is a general

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factor with $B_v = \{1, 3, 5, \ldots, f(v)\}$, and a $(1, f)$-odd subgraph is a general factor with $B_v = \{0, 1, 3, 5, \ldots, f(v)\}$.

The general problem becomes NP-complete if B_v may have gaps more then 1 ([5], [2]), so restrict B_v to sets with gaps at most 1. If there is no general factor in G, we can define an optimal solution in the following sense. Let the *deficiency* of a spanning subgraph F at a vertex v be defined as the distance between the degree of v in F and the set B_v. Specifically, let l_v and u_v be the smallest and largest elements of B_v. Then the deficiency of F at v is

$$\mathrm{def}_F(v) = \begin{cases} 0, & \text{if } \deg_F(v) \in B_v \\ l_v - \deg_F(v), & \text{if } \deg_F(v) < l_v \\ \deg_F(v) - u_v, & \text{if } \deg_F(v) > u_v \\ 1, & \text{if } \deg_F(v) \notin B_v \text{ and } l_v < \deg_F(v) < u_v, \end{cases}$$

since there are no gaps of length 2. The *total deficiency* of F is defined as

$$\mathrm{def}(F) = \sum_{v \in V(G)} \mathrm{def}_F(v).$$

We say that F is *optimal* if it has the smallest total deficiency.

Applying this definition to our special case, by setting $B_v = \{1, 3, 5, \ldots, f(v)\}$ for each v, we obtain the definition of an *optimal* $(1, f)$*-odd spanning subgraph*. Note that this definition is different from the above definition of a maximum $(1, f)$-odd subgraph. In fact, it is not true that the edge set of every optimal $(1, f)$-odd spanning subgraph induces a real $(1, f)$-odd subgraph. On the other hand, we will show in Section 2 that there exists an optimal $(1, f)$-odd spanning subgraph whose edge set induces a maximum $(1, f)$-odd subgraph. This means that the total deficiency of any optimal $(1, f)$-odd spanning subgraph is equal to the number of uncovered vertices in a maximum $(1, f)$-odd subgraph.

In Section 3 a Gallai-Edmonds type structure theorem is given. This result does not seems to be easily deducible from Lovász's [5] theorems, so we prove it directly.

The usual method to give an algorithm for these kind of problems is to construct a graph G' from the given graph G such that the order of a maximum $(1, f)$–odd subgraph in G is a function of the order of a maximum matching in G'. Unfortunately such a construction is not known for this problem.

The algorithm in [2] is an Edmonds type algorithm, but it does not seems to be a generalisation of Edmonds blossom algorithm. Its running time is $O(|V(G)|^5)$. The polynomial algorithm given in Section 4 follows a different approach from this. Our algorithm is a direct generalisation of Edmonds blossom algorithm with running time $O(|V(G)|^3)$.

2 Maximum is Optimal

For problems of finding a maximum $(1, f)$–odd subgraph and an optimal $(1, f)$–odd spanning subgraph, the number of edges in the solution is not determined by the number of vertices, that is, there can be several solutions with different numbers of edges. However, we can make some useful observations about the solutions with minimum number of edges. Let us call a maximum $(1, f)$–odd subgraph with the fewest edges a *smallest maximum* $(1, f)$*–odd subgraph*, and call an optimal $(1,f)$–odd spanning subgraph with the fewest edges a *smallest optimal* $(1,f)$*–odd spanning subgraph.*

Observation 1 *A smallest maximum* $(1,f)$*–odd subgraph is a forest.*

PROOF: Suppose indirectly, that a smallest maximum $(1,f)$–odd subgraph F contains a cycle. By removing all edges of this cycle, the degrees of vertices in F remain odd, hence cannot decrease to 0. Thus the new subgraph covers the same vertices as F, but it has fewer edges, contradicting our assumption. □

Theorem 2 *The edge set of a smallest optimal* $(1,f)$*–odd spanning subgraph induces a maximum* $(1,f)$*–odd subgraph.*

PROOF: First note that the edge set of a smallest optimal $(1,f)$–odd spanning subgraph does not necessarily induce a $(1,f)$–odd subgraph since it may contain vertices v with $\deg_F(v) \notin B_v \cup \{0\}$ where $B_v = \{1,3,\ldots,f(v)\}$.

Let F be a smallest optimal $(1,f)$–odd spanning subgraph for which the number of vertices v with $\deg_F(v) \notin B_v \cup \{0\}$ is minimum. If there is no such vertex, then $E(F)$ induces a maximum $(1,f)$–odd subgraph. Otherwise, let x be a vertex such that $\deg_F(x) \notin B_x \cup \{0\}$. If we remove any edge xy, then $\mathrm{def}(x)$ decreases by 1, and $\mathrm{def}(y)$ is either increased or decreased by 1. Thus $\mathrm{def}(F)$ is either decreased by 2 or does not change. So, remove some edges incident to x to obtain F' until its degree will be in B_x. If $\mathrm{def}(F') < \mathrm{def}(F)$ then we have a contradiction since F is an optimal $(1,f)$–odd spanning subgraph. Otherwise, F' is an optimal $(1,f)$–odd spanning subgraph, as well, but it has fewer edges than F, which contradicts choice of F. Therefore, the edge set of a smallest optimal $(1,f)$-odd spanning subgraph induces a $(1,f)$-odd subgraph, which does not cover exactly $\mathrm{def}(F)$ vertices and must be a maximum $(1,f)$-odd subgraph. □

3 Structure Theorem

For subsets A and B of a set, we denote by $A \subseteq B$ if A is a subset of B, and by $A \subset B$ if A is a proper subset of B. Let G be a graph. A component of G is called an *odd component* if it has an odd order, and the number of odd components of G is denoted by $\mathrm{odd}(G)$. A subset $X \subseteq V(G)$ is called a **barrier** in G for $(1,f)$-odd factor if

$$\mathrm{barr}(G) = \max_{S \subseteq V(G)} \{\mathrm{odd}(G-S) - \sum_{x \in S} f(x)\} = \mathrm{odd}(G-X) - \sum_{x \in X} f(x). \tag{1}$$

where $\mathrm{barr}(G)$ is defined as above and $\mathrm{barr}(G) \geq 0$ by setting $S = \emptyset$. A barrier X is said to be **minimal** if no proper subset of X is a barrier.

Theorem 3 (Cui, Kano [3]) *A graph G contains a $(1,f)$-odd factor if and only if* $\mathrm{barr}(G) = 0$*, that is, if and only if* $\mathrm{odd}(G-S) \leq \sum_{x \in S} f(x)$ *for all* $S \subseteq V(G)$*.*

Theorem 4 ([4]) *The order* $|H|$ *of a maximum* $(1,f)$*–odd subgraph* H *of a graph* G *is given by*

$$|H| = |G| - \mathrm{barr}(G).$$

Theorem 5 (Topp, Vestergaard [7]) *Let* G *be a graph having no* $(1,f)$*-odd factors. Let* X *be a minimal barrier for* $(1,f)$*-odd factor in* G*. Then every vertex* $v \in X$ *is adjacent to at least* $f(v)+2$ *odd components of* $G-X$*. In particular, there exists a subset* $A_v \subseteq V(G)$ *such that* $\langle A_v \rangle_G = K_{1,f(v)+2}$ *and its center is* v*.*

Proof: Suppose that a vertex $v \in X$ is adjacent to at most $f(v)+1$ odd components of $G-X$. If v is adjacent to exactly $f(v)+1$ odd components of $G-X$, then v and these $f(v)+1$ odd components form a new odd components of $G-(X-v)$. Hence, in any case, we have

$$\mathrm{odd}(G-(X-v)) \geq \mathrm{odd}(G-X) - f(v).$$

On the other hand, by (1), we have

$$\mathrm{odd}(G-(X-v)) - \sum_{x \in X-v} f(x) \leq \mathrm{odd}(G-X) - \sum_{x \in X} f(x).$$

Thus $\mathrm{odd}(G-(X-v)) = \mathrm{odd}(G-X) - f(v)$, and so

$$\mathrm{odd}(G-(X-v)) - \sum_{x \in X-v} f(x) = \mathrm{odd}(G-X) - \sum_{x \in X} f(x),$$

which implies that $X-v$ is also a barrier. This contradicts the minimality of X. Therefore v adjacent to at least $f(v)+2$ odd components of $G-X$. The latter part follows immediately from the former part. □

For a graph G, define

$$\tau(G) = \text{the order of a maximum } (1,f)\text{-odd subgraph of } G.$$

For any vertex x of G, we denote by G_x the graph obtained from G by adding a new vertex w together with a new edge wx (see Figure 1 **Please add such a figure in it**). Let $D(G)$ denote the set of all vertices x of G such that $\tau(G_x) = \tau(G)+2$. Let $A(G)$ be the set of vertices of $V(G)-D(G)$ that are adjacent to at least one vertex in $D(G)$. Finally, define $C(G) = V(G) - D(G) - A(G)$. Then $V(G)$ is decomposed into three disjoint subsets

$$V(G) = D(G) \cup A(G) \cup C(G). \tag{2}$$

Note that if $f(x) = 1$ for all vertices x of G, then a maximum $(1,f)$-odd subgraph is a maximum matching and a vertex y satisfies $\tau(G_y) = \tau(G)+2$ if and only if y is not contained in a certain maximum matching in G, and thus the above decomposition $V(G) = D(G) \cup A(G) \cup C(G)$ becomes the Gallai-Edmonds decomposition ([6] p.94).

A graph G is said to be **critical with respect to** $(1,f)$**-odd factor** if for every vertex x of G, $G_x = G + wx$ has a $(1,f)$-odd factor, where $f(w)$ is defined to be one. It is obvious that if G is critical with respect to $(1,f)$-odd factor, then G is a connected graph of odd order.

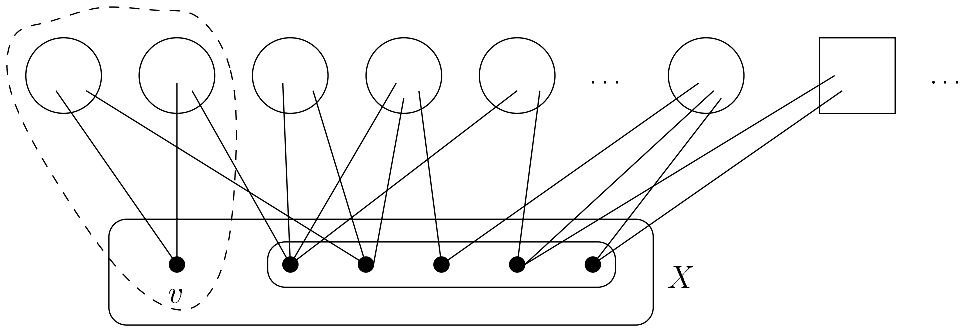

Figure 1: Components of $G - X$ and those of $G - (X - v)$.

Theorem 6 (Structure Theorem on $(1, f)$-odd subgraphs) *Let G be a graph, and $V(G) = D(G) \cup A(G) \cup C(G)$ the decomposition defined in (2). Then the following statements hold (see Figure 2):*

(i) Every component of $\langle D(G)\rangle_G$ is critical with respect to $(1, f)$-odd factor.

(ii) $\langle C(G)\rangle_G$ has a $(1, f)$-odd factor.

(iii) Every maximum $(1, f)$-odd subgraph H of G covers $C(G) \cup A(G)$, and for every vertex $x \in A(G)$, $\deg_H(x) = f(x)$ and every edge of H incident with x joins x to a vertex in $D(G)$.

(iv) The order $|H|$ of a maximum $(1, f)$-odd subgraph H is given by

$$|H| = |G| + \omega(\langle D(G)\rangle_G) - \sum_{x \in A(G)} f(x), \tag{3}$$

where $\omega(\langle D(G)\rangle_G)$ denotes the number of components of $\langle D(G)\rangle_G$.

Proof: We may assume that G is connected since if the theorem holds for each component of G, then the theorem holds for G. Moreover, we may assume that G has no $(1, f)$-odd factor since otherwise $D(G) = \emptyset$, $A(G) = \emptyset$ and $C(G) = V(G)$, and thus the theorem holds.

Let S be a maximal barrier of G, that is, S is a subset of $V(G)$ such that

$$\mathrm{odd}(G - S) - \sum_{x \in S} f(x) = \max_{X \subset V(G)} \{\mathrm{odd}(G - X) - \sum_{x \in X} f(x)\} = \mathrm{barr}(G) > 0 \tag{4}$$

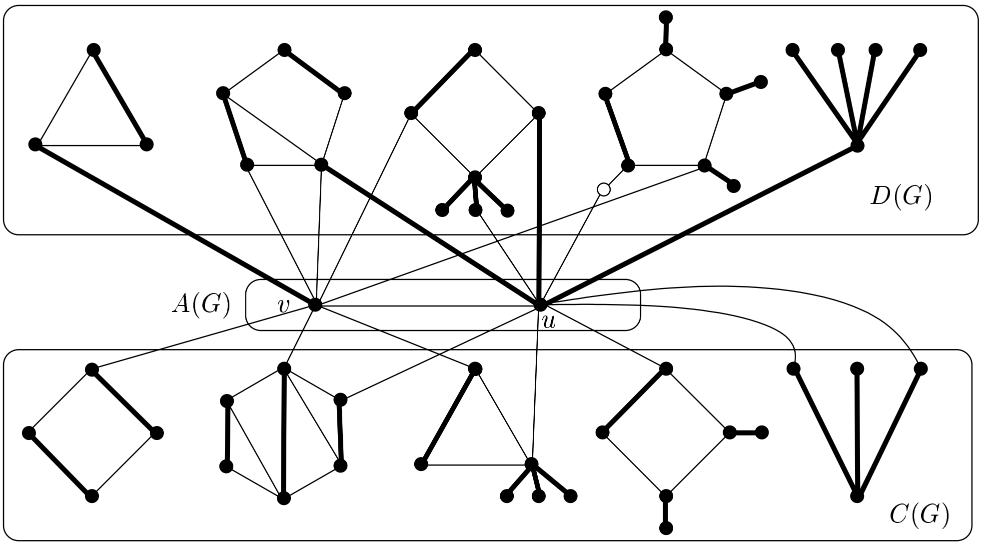

Figure 2: Decomposition $V(G) = D(G) \cup A(G) \cup C(G)$, with $f(v) = 1, f(u) = 3$ and one uncovered vertex.

and

$$\text{odd}(G - T) - \sum_{x \in T} f(x) < \text{odd}(G - S) - \sum_{x \in S} f(x) \quad \text{for all} \quad S \subset T \subseteq V(G). \tag{5}$$

Claim 1 *Every even component of $G - S$ has a $(1, f)$-odd factor.*

Proof: Assume that an even component D of $G - S$ does not have a $(1, f)$-odd factor. Then by Theorem 3, there exists a subset $\emptyset \neq X \subset V(D)$ such that $\text{odd}(D - X) > \sum_{x \in X} f(x)$. Then

$$\begin{aligned} \text{odd}(G - (S \cup X)) - \sum_{x \in S \cup X} f(x) &= \text{odd}(G - S) - \sum_{x \in S} f(x) + \text{odd}(D - X) - \sum_{x \in X} f(x) \\ &> \text{odd}(G - S) - \sum_{x \in S} f(x), \end{aligned}$$

contrary to (4). Hence D has a $(1, f)$-odd factor. □

Claim 2 *Every odd component of $G - S$ is critical with respect to $(1, f)$-odd factor.*

PROOF: Suppose that an odd component C of $G - S$ is not critical with respect to $(1, f)$-odd factor. Then there exist a vertex $v \in V(C)$ such that C_v has no $(1, f)$-odd factor. Let Y be a minimal barrier of $C_v = C + vw$. Then w is not contained in Y by Theorem 5, and so $\emptyset \neq Y \subseteq V(C)$. It follows that

$$\mathrm{odd}(C - Y) \geq \mathrm{odd}(C + vw - Y) - 1 \geq \sum_{x \in Y} f(x) + 2 - 1 = \sum_{x \in Y} f(x) + 1.$$

Hence

$$\begin{aligned} &\mathrm{odd}(G - (S \cup Y)) - \sum_{x \in S \cup Y} f(x) \\ &= \mathrm{odd}(G - S) - 1 + \mathrm{odd}(C - Y) - \sum_{x \in S} f(x) - \sum_{x \in Y} f(x) \\ &\geq \mathrm{odd}(G - S) - \sum_{x \in S} f(x), \end{aligned}$$

which implies that $S \cup Y$ is a barrier in G. This contradicts the maximality (5) of S. Hence C is critical with respect to $(1, f)$-odd factor. $\square$

Let $\{C_1, C_2, \ldots, C_m\}$, where $m = \mathrm{odd}(G - S) > \sum_{x \in S} f(x)$, be the set of odd components of $G - S$. We define the bipartite graph B with bipartite sets S and $\{C_1, C_2, \ldots, C_m\}$ as follows: a vertex $x \in S$ and C_i is joined by an edge of B if and only if x and C_i are joined by at least one edge of G. In other words, each of $\{C_1, C_2, \ldots, C_m\}$ is contracted to one vertex and all edges inside S are removed.

Claim 3 *$|N_B(X)| \geq \sum_{x \in X} f(x)$ for all $X \subseteq S$.*

PROOF: Assume that $|N_B(Y)| < \sum_{x \in Y} f(x)$ for some $\emptyset \neq Y \subseteq S$. Then

$$\begin{aligned} &\mathrm{odd}(G - (S - Y)) - \sum_{x \in S - Y} f(x) \geq |\{C_1, C_2, \ldots, C_m\} - N_B(Y)| - \sum_{x \in S - Y} f(x) \\ > \ &m - \sum_{x \in Y} f(x) - \sum_{x \in S - Y} f(x) = m - \sum_{x \in S} f(x) = \mathrm{odd}(G - S) - \sum_{x \in S} f(x), \end{aligned}$$

contrary to (4). Hence Claim 3 holds. $\square$

Claim 4 *There exists the unique maximum proper subset $S_0 \subset S$ in B such that $|N_B(S_0)| = \sum_{x \in S_0} f(x)$. Furthermore, $|N_B(Y) \setminus N_B(S_0)| > \sum_{x \in Y} f(x)$ for every $\emptyset \neq Y \subseteq S - S_0$.*

PROOF: It follows from (4) that $|N_B(S)| = m > \sum_{x \in S} f(x)$. Suppose that $|N_B(X_1)| = \sum_{x \in X_1} f(x)$ and $|N_B(X_2)| = \sum_{x \in X_2} f(x)$ for two subsets X_1, $X_2 \subset S$. Then by Claim 3, we have

$$\begin{aligned} \sum_{x \in X_1 \cup X_2} f(x) &\leq |N_B(X_1 \cup X_2)| \leq |N_B(X_1)| + |N_B(X_2)| - |N_B(X_1 \cap X_2)| \\ &\leq \sum_{x \in X_1} f(x) + \sum_{x \in X_2} f(x) - \sum_{x \in X_1 \cap X_2} f(x) = \sum_{x \in X_1 \cup X_2} f(x) \end{aligned}$$

Hence $|N_B(X_1 \cup X_2)| = \sum_{x \in X_1 \cup X_2} f(x)$. Therefore there exists the unique maximum subset $S_0 \subset S$ such that $|N_B(S_0)| = \sum_{x \in S_0} f(x)$.

Let $\emptyset \neq Y \subseteq S - S_0$. Then it follows from the maximality of S_0 and $S_0 \subset Y \cup S_0$ that

$$|N_B(Y) \setminus N_B(S_0)| = |N_B(Y \cup S_0) - N_B(S_0)| > \sum_{x \in Y \cup S_0} f(x) - \sum_{x \in S_0} f(x) = \sum_{x \in Y} f(x).$$

Therefore the claim is proved. $\square$

Let $T = S - S_0$. Then

$$\begin{aligned} \operatorname{odd}(G-T) &\geq |\{C_1, C_2, \ldots, C_m\} - N_B(S_0)| = m - \sum_{x \in S_0} f(x) \\ &= \operatorname{odd}(G-S) - \left(\sum_{x \in S} f(x) - \sum_{x \in T} f(x) \right), \end{aligned}$$

and so

$$\operatorname{odd}(G-T) - \sum_{x \in T} f(x) = \operatorname{odd}(G-S) - \sum_{x \in S} f(x) = \operatorname{barr}(G).$$

Let $\{C_1, C_2, \ldots, C_k\}$ be the set of odd components of $G - T$, where $k = \operatorname{odd}(G-T)$, and $\{C'_1, C'_2, \ldots, C'_r\}$ the set of odd components of $G - S$ which corresponds to $N_B(S_0)$, where $r = |N_B(S_0)| = \sum_{x \in S_0} f(x)$. Then by Hall's Marriage Theorem and by Claims 3 and 4, it follows that

(i) B has a subgraph M satisfying

$$\begin{aligned} &\deg_M(x) = f(x) \quad \text{for all } x \in S, \\ &\deg_M(C') = 1 \quad \text{for all } C' \in \{C'_1, C'_2, \ldots, C'_r\}, \quad \text{and} \\ &\deg_M(C) = 1 \quad \text{for all } C \in \{C_1, C_2, \ldots, C_k\} \cap V(M). \end{aligned}$$

(ii) for each C_i $(1 \leq i \leq k)$, there exist subgraphs M_1 and M_2 of B satisfying the above condition (i) such that C_i is covered by M_1 but not by M_2.

Let M be a subgraph of B given in (i). Then for every odd component C'_j $(1 \leq j \leq r)$, there exists an edge in M joining C'_j to a vertex x_j in S. Take an edge e_j of G joining x_j to a vertex v_j in C'_j, and let R'_j be a $(1, f)$-odd factor of $C'_j + v_j x_j$, whose existence is guaranteed by Claim 2.

For an odd component C_i $(1 \leq i \leq m)$, if M has an edge joining C_i to a vertex x_i of S, then there exists an edge e_i of G joining x_i to a vertex v_i of C_i, and take a $(1, f)$-odd factor R_i of $C_i + v_i x_i$. If M has no such an edge, then take a maximum $(1, f)$-odd subgraph H_i of C_i, whose order is $|C_i| - 1$ by Claim 2. Define

$$\begin{aligned} K = &\bigcup \{(1,f)\text{-odd factor of even components of } G - S\} \\ &+ \bigcup_{1 \leq j \leq r} \{R'_j\} + \bigcup_{1 \leq i \leq k} \{R_i \text{ or } H_i\}. \end{aligned}$$

Then for each vertex $x \in S$, it follows that $\deg_K(x) = \deg_M(x) = f(x)$ by (i), and by Theorem 4, K is a maximum $(1, f)$-odd subgraph of G since the order of K is $|G| - (k - \sum_{x \in T} f(x)) = |G| - \mathrm{barr}(G)$.

Conversely, every maximum $(1, f)$–odd subgraph in G is obtained in this way since for any maximum $(1, f)$–odd subgraph H in G, H cannot cover at least $k - \sum_{x \in T} f(x)$ odd components in $\{C_1, C_2, \dots, C_k\}$, and at least one of whose vertices is not contained is H. Since H is a maximum $(1, f)$–odd subgraph, H does not cover exactly $k - \sum_{x \in T} f(x)$ of these vertices and covers all the other vertices. Therefore H induces a subgraph N in B which covers S and $\{C'_1, C'_2, \dots, C'_r\}$ and $k - \sum_{x \in T} f(x)$ elements in $\{C_1, C_2, \dots, C_k\}$, and thus N satisfies the condition (i) as M does. Therefore H can be constructed from a subgraph of B satisfying (i) in the same way as K is obtained.

Claim 5 *$D(G) = V(C_1) \cup V(C_2) \cup \cdots \cup V(C_k)$ and $A(G) = T$.*

PROOF: It is clear that for every vertex x of any C_t $(1 \le t \le k)$, B has a subgraph that satisfies (i) and does not cover x by (ii). Since $C_t - x$ has a $(1, f)$–odd factor by Claim 2, there exists a maximum $(1, f)$–odd subgraph in G which does not cover x. Thus $V(C_1) \cup V(C_2) \cup \cdots \cup V(C_k) \subseteq D(G)$.

Since every maximum $(1, f)$–odd subgraph in G is obtained in the way mentioned above, $D(G) \subseteq V(C_1) \cup V(C_2) \cup \cdots \cup V(C_k)$. Consequently, $D(G) = V(C_1) \cup V(C_2) \cup \cdots \cup V(C_k)$.

Since $N_B(S_0) = \{C'_1, C'_2, \dots, C'_r\}$, it follows that $N_B(\{C_1, C_2, \dots, C_k\}) = S - S_0 = T$. Therefore

$$A(G) = N_G(V(C_1) \cup \cdots \cup V(C_k)) \setminus (V(C_1) \cup \cdots \cup V(C_k)) = T.$$

Hence the claim is proved. □

It is easy to see that $\langle V(C'_1) \cup \cdots \cup V(C'_r) \cup S_0 \rangle$ has a $(1, f)$-odd factor, and forms even components of $G - T$. Consequently, (i)-(iv) are proved, and the proof of Theorem 6 is complete. □

Since every component of $\langle D(G) \rangle_G$ is factor critical, the Gallai-Edmonds Structure Theorem on matchings is an easy consequence of the above Structure Theorem.

A subgraph H of a graph G is said to *avoid* a subset $X \subseteq V(G)$ if H contains no vertex in X. The following theorem, which holds for matching ([6] p.88), is conjectured in [4].

Theorem 7 *Let G be a graph, and X and Y be two subsets of $V(G)$ such that $|X| < |Y|$. If there exist a maximum $(1, f)$-odd subgraph which avoids X and one which avoids Y, then there exists a maximum $(1, f)$-odd subgraph which avoids X and at least one vertex of $Y \setminus X$.*

PROOF: Let H_X and H_Y be maximum $(1, f)$-odd subgraphs which avoid X and Y, respectively. We may assume that H_X covers all the vertices in $Y \setminus X$ since otherwise H_X is the desired maximum $(1, f)$–odd subgraph. Let $\{C_1, C_2, \dots, C_k\}$ be the components of $\langle D(G) \rangle_G$, and B denote the bipartite subgraph with bipartite sets T and $\{C_1, C_2, \dots, C_k\}$ defined in the proof of Theorem 6. By Theorem 6, X and Y consist of vertices that are taken from each C_i at most one.

By Theorem 6 and by $|X| < |Y|$, there exist a vertex $v \in A(G)$ and two components C_s and C_t in $\{C_1, C_2, \ldots, C_k\}$ such that both C_s and C_t are adjacent to v in G, H_X covers C_s but not C_t, $V(C_s) \cap X = \emptyset$, $V(C_s) \cap Y \neq \emptyset$ and $V(C_t) \cap (X \cup Y) = \emptyset$. Then by removing the edge joining v to C_s of the subgraph in B corresponding to H_X, and by adding an edge of B joining v to C_t, we obtain a new subgraph of B satisfying the condition (i) given in the proof of Theorem 6, and can construct a maximum $(1,f)$–odd subgraph H of G from it as in the proof of Theorem 6, which covers C_t but not the unique vertex in $V(C_s) \cap Y$ and avoids X. Therefore H is the desired subgraph, and the theorem is proved. □

4 Augmenting walks

For subgraphs H and K of a graph G, we denote by $H \triangle K$ the subgraph of G induced by $E(H) \triangle E(K) = (E(H) \cup E(K)) - (E(H) \cap E(K))$.

Let F be a $(1,f)$–odd subgraph in G. Edges of F will be called **blue edges** and edges of $G - F$ are called **red edges**. For a subgraph S of G and a vertex v of S, $\deg_S^B(v)$ ($\deg_S^R(v)$) denotes the number of blue (red) edges incident with v in S. In particular, $\deg_F(v) = \deg_G^B(v)$. A vertex v is **saturated** if $\deg_F(v) = f(v)$.

An F–**augmenting walk** between x and y is a walk W such that

(i) $\deg_W^B(x) = \deg_W^B(y) = 0$,

(ii) $\deg_W^R(x) = \deg_W^R(y) = 1$,

(iii) $\deg_W^R(v) - \deg_W^B(v) \leq f(v) - \deg_F(v) \qquad$ for all $v \in V(W) - \{x, y\}$.

A $(1,f)$-odd subgraph H of G is said to be *maximal* if G has no $(1,f)$-odd subgraph H' such that $V(H) \subset V(H')$. Obviously, a maximum $(1,f)$–odd subgraph is a maximal $(1,f)$-odd subgraph. The next lemma was directly proved in [4], and can be easily shown by using Theorem 6.

Lemma 8 *A maximal $(1,f)$–odd subgraph is a maximum $(1,f)$–odd subgraph.*

The following lemma will not be used directly to prove the correctness of the algorithm, however it helps to understand the concept of the algorithm. On the other hand, the algorithm will provide a stronger result about the properties of augmenting walks in a non-maximum $(1,f)$–odd subgraph.

Lemma 9 *A $(1,f)$–odd subgraph is maximum if and only if there is no augmenting-walk.*

PROOF: Suppose that F is a $(1,f)$–odd subgraph of G and there is an F–augmenting-walk W between x and y, where $x, y \notin V(F)$. Then $W \triangle F$ is a $(1,f)$-odd subgraph since for every vertex $v \in V(W) - \{x, y\}$,

$$\deg_{W \triangle F}(v) = \deg_F(v) - \deg_W^B(v) + \deg_W^R(v) \equiv \deg_F(v) \pmod 2$$

and $\deg_{W\triangle F}(v) \leq f(v)$ by (iii). Furthermore, $W\triangle F$ covers all the vertices of F and $\{x, y\}$, therefore F cannot be maximum.

Suppose now that a $(1, f)$–odd subgraph F of G is not maximum. By Lemma 8, there exists a $(1, f)$–odd subgraph F' such that $V(F) \subset V(F')$. Let $H = F\triangle F'$. We call edges in $E(H) \cap E(F')$ red edges, and edges in $E(H) \cap E(F)$ blue edges. Then H has the following properties.

(iv) $\deg_H^B(x) = 0 \quad$ for all $x \in V(F') - V(F)$,

(v) $\deg_H^R(x)$ is positive and odd for all $x \in V(F') - V(F)$,

(vi) $\deg_H^R(v) - \deg_H^B(v) \leq f(v) - \deg_F(v)$ and $\deg_H^R(v) \pm \deg_H^B(v)$ is even for all $v \in V(F)$.

This may not be the desired augmenting-walk. We start to build a walk from an arbitrary vertex of $V(F') - V(F)$, say x. If x has a neighbour in $V(F') - V(F)$ in H, then this red edge is an augmenting-walk. Otherwise, we choose any red edge and its endvertex in $V(F)$ to continue the walk. When the next edge is selected to go further only one rule has to be applied. If it is possible then red edges and blue edges are used alternately, otherwise we continue on an edge of the same colour. So, if we arrive to a vertex u through red edge and there is an unused blue edge incident with u then we continue on this blue edge and vice versa.

Since all vertices of $V(F)$ have even degree in H, this path will reach a vertex of $V(F')-V(F)$. If this vertex is different from x then we obtain the desired augmenting-walk. Properties (i) and (ii) clearly hold. The validity of (iii) follows from (vi) and the rule of the construction of the walk.

If we get to x again, then deleting the edges of this closed walk keeps the properties of H, so the existence of the desired walk can be proved by induction on the number of edges incident with vertices of $V(F') - V(F)$. $\square$

Therefore, if we can give an algorithm which finds an augmenting-walk, then we can construct an algorithm to find a maximum $(1, f)$–odd subgraph in a natural way.

5 The algorithm

The algorithm is a generalisation of Edmonds's algorithm for finding a maximum matching. We will adopt the terminology and notation used in [1] to describe the Edmonds algorithm.

The main idea is to search for augmenting-paths (augmenting-walks with maximum degree 2) with a Breadth First Search style method. If no augmenting-path is found then try to shrink some cycles. If this fails, too, then we have a maximum $(1, f)$–odd subgraph.

We define a basic structure maintained by the algorithm. Suppose we have a $(1, f)$–odd subgraph F of G which is a forest and whose edges are called blue edges. The vertices of $V(F)$ are said to be F-*covered* and the vertices of $V(G) - V(F)$ are said to be F-*exposed*. Recall that a vertex v of F is said to be *saturated* if $\deg_F(v) = f(v)$. Let r be a fixed F–exposed vertex. We build up a rooted tree T with root r, where the root lies on the top and its ancestors lie below, such that red edges connect r and some components of F to each other forming a tree. This tree will satisfy the following properties (see Fig. 3):

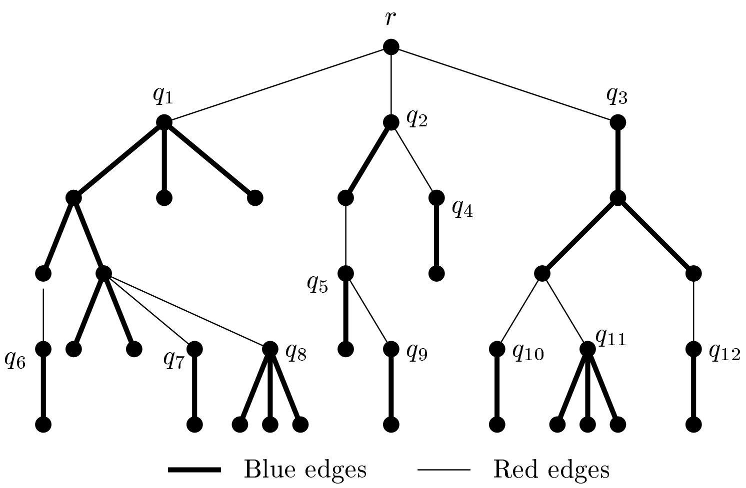

Figure 3: Alternating tree

a) The blue degree of r is even (initially zero).

b) The blue degree of every $v \in V(T) - \{r\}$ is odd and therefore > 0.

c) If T contains a vertex v of F, then it contains the whole component of F containing v.

d) If a vertex $v \in V(T) - \{r\}$ is saturated, then no red edge goes downward from v, but it may happen that a red edge goes upward from v.

This tree is called an F–**alternating tree** for similarity to [1]. These properties guarantee that the path from any vertex to r is an F–augmenting-path. If T contains a component T_i of F then let q_i denote the unique vertex of T_i which is closest to r. We also maintain a variable $s(t)$ for each vertex $t \in T$:

$$s(t) = \begin{cases} SAT, & \text{if } t = q_i \text{ for some } i \text{ and } t \text{ is saturated} \\ UNSAT, & \text{if } t = q_i \text{ for some } i \text{ and } t \text{ is unsaturated} \\ NON, & \text{otherwise} \end{cases}$$

It seems that we loose information by forgeting the actual f values, but in one augmentation step we will not increase the degrees of the $(1, f)$-odd subgraph by more that 2 in any unsaturated q_i, and we will not increase it at all in other vertices. After the augmentation step the function s is recalculated from the original f.

Let $A(T)$ denote the subset of all vertices t for which $s(t) = SAT$, and let $B(T) = V(T) - A(T)$. In the algorithm, basic subroutines will be used which corresponds to extension, augmentation and shrinking in the Edmonds algorithm.

Use vw to extend T

Input: A $(1, f)$–odd subgraph F of G, an F–alternating tree T, and an edge vw of G such that $v \in B(T)$, $w \notin V(T)$ and w is F–covered.

Action: Let T_i be the component of F containing w. (Note that non of its vertices are in $V(T)$.) Replace T by the tree having edge–set $E(T) \cup \{vw\} \cup E(T_i)$, set $q_i = w$ and set $s(w)$ according to the relation of $f(w)$ and $\deg_F(w)$. For any other vertex $u \in V(T_i)$ set $s(u) = NON$.

It is trivial that this step maintains the properties of the F–alternating tree. It is used to extend some of the alternating-paths.

Use vw to augment F'

Input: A $(1, f)$–odd subgraph F of G, an F–alternating tree T, and edge vw of G such that $v \in B(T)$, $w \notin V(T)$ and w is F–exposed.

Action: Let P be the path obtained by attaching vw to the path from r to v in T. (Note that P is an F–augmenting-walk.) Replace F by $F' = F \triangle P$.

In this step a larger $(1, f)$–odd subgraph is found, we will start to build a new alternating tree after this.

For the shrinking step a new notation is needed. Also some extra manipulation is necessary when the length of the cycle is even. Let $G' = G/C$ be the graph obtained from G by shrinking C, as follows. Let $V(C) = \{k_1, k_2, \ldots, k_l\}$. If the length l of C is odd, then G/C has vertex set $(V(G) - V(C)) \cup \{c\}$, the edge set is obtained by deleting all edges spanned by $V(C)$; all vertices of $V(C)$ are identified with the new vertex c, so all edges with one end on the cycle will have that end at c. This new vertex c is called a **pseudo-vertex**.

If the length of C is even, then first we shrink C in the same way as above, then we attach an extra vertex c' to c with a new blue edge cc'. So $V(G') = (V(G) - V(C)) \cup \{c, c'\}$, $E(G') = E(G) - E(C) \cup \{c, c'\}$ and $\deg_{G'}(c') = 1$. This extra edge is called a **dummy edge**. The variables $s(c)$ and $s(c')$ are defined in the following procedure.

Use vw to shrink a cycle

Input: A $(1, f)$–odd subgraph F of G, an F–alternating tree T, and edge vw of G such that $v, w \in B(T)$.

Action: Let C be the cycle formed by vw together with the path in T from v to w, and let z be the unique vertex of C on the highest level of the tree. If the length of C is odd then replace G by $G' = G/C$, F by $F' = F \setminus E(C)$ and T by the tree T' of G' having edge-set $E(T) - E(C)$. If the length of C is even then do the same, but add the dummy edge cc' to F' and T', too.

Finally set $s(c) := s(z)$, but reset $s(c) = UNSAT$ if both edges of C incident with z are blue, and set $s(c') = NON$ in the appropriate case.

Lemma 10 *The tree T' obtained in the above step is an F'-alternating tree in G'.*

Proof: To prove that properties a) and b) hold, it must be shown that $\deg^B_{T'}(c) - \deg^B_T(z)$ is even. Let us first deal with the case when the length of C was odd and $z \neq r$.

In this case the sum of the blue degrees of all vertices of C is odd, since each of them is odd. The blue edges of C are counted twice in this sum. Thus $\deg^B_{T'}(c)$ is odd since it is the number of edges incident with a vertex on C minus twice the number of blue edges of C. Hence $\deg^B_{T'}(c) - \deg^B_T(z)$ is even, because $\deg^B_T(z)$ is also odd.

In the other case, when C is an even cycle and $z \neq r$, a similar argument shows that $\deg^B_{T'}(c)$ is odd, since we have attached the dummy-edge to c. Hence $\deg^B_{T'}(c) - \deg^B_T(z)$ is even again.

If $z = r$ then the proof works in a similar way.

Property c) trivially holds. To prove that property d) holds, we show that $s(c) \neq SAT$. The only way to obtain $s(c) = SAT$ is that $s(z) = SAT$. However, in this case no red edge goes downwards from z, therefore both edges of C incident to z must be blue. But then $s(c)$ must be set to $UNSAT$. $\square$

After this step we continue with several other extension and shrinking steps, until we find an augmenting-path. We need to show now, that if there is an augmenting-walk in the shrinked graph then there is one in the original graph. Unfortunately, this is not true in general, only for "elementary augmenting-walks".

An augmenting-walk W is called *elementary* if for every vertex v of the walk $\deg^R_W(v) \leq 2$, $\deg^B_W(v) \leq 2$ and if $\deg^R_W(v) = 2$ then it contains the edge connecting v to its parent. This implies that for any internal vertex v of the walk either

a) $\deg^R_W(v) = \deg^B_W(v) = 1$ or

b) $\deg^R_W(v) = 0, \deg^B_W(v) = 2$ or

c) $\deg^R_W(v) = \deg^B_W(v) = 2$ or

d) $\deg^R_W(v) = 2, \deg^B_W(v) = 0$, but this is only possible if $s(v) = UNSAT$.

The algorithm might do a few shrinking steps until it finds a augmenting-walk in the actual alternating-tree. Fortunately, an augmenting-walk in an alternating-tree is always elementary, so by the next Lemma, if we can augment in the shrinked graph, we can "blow up" the augmenting-walk into an elementary augmenting-walk in the original, unshrinked graph.

It is enough to show the following for one shrinking step.

Lemma 11 *Let F be a $(1,f)$-odd subgraph of G and let $G' = G/C$ and F' be the $(1,f)$-odd subgraph of G' obtained during the shrinking procedure from F. If G' contains an elementary F'-augmenting-walk W', then G contains an elementary F-augmenting-walk W.*

PROOF: We consider the following cases and subcases:

Case i) *c is not the root of G', and the length of C is odd.*

a) $\deg^R_{W'}(c) = 1$, $\deg^B_{W'}(c) = 1$
This means that W' contains one blue and one red edge incident to c in the shrinked graph. In the original graph these two edges have one end on the cycle C. Let k_i and k_j be the vertices of the cycle which are the endvertices of the red and blue edges, respectively. If $i = j$ then let $W = W'$, it is clear that this is elementary, too. If $i \neq j$ then W' is broken into two parts. These two parts can be connected using the edges of C in the following way.

If at least one of the edges of C incident to k_i is blue, then take that arc of C connecting k_i to k_j which starts with this blue edge. We have to show that the resulting walk is an elementary augmenting-walk.

The conditions are surely satisfied outside of C. It is also satisfied in k_i since $\deg^R_W(k_i) = 1$, $\deg^B_W(k_i) = 1$. For k_j we have $\deg^B_W(k_i) \geq 1$, so the conditions hold regardless of the colour of the other end of the arc.

In the other vertices of the arc, there is only one way to violate the augmenting-walk conditions: if there are two red edges incident to a saturated vertex. However, this is not possible, since saturated vertices in the alternating tree cannot have two red edges incident with them.

If there are two red edges incident to an unsaturated vertex then it will be an augmenting-walk, but we have to show that it is elementary. The only nontrivial part is that one of the red edges must be the edge connecting the vertex to its parent. This follows from the procedure of obtaining C. All but one edge of C are edges of the augmenting tree, so they connect a vertex to its parent. The exceptional edge is the one which "closed" the cycle before shrinking.

In the other case, when both edges of C incident to k_i are red, take any of the two arcs connecting k_i to k_j. By the previous argument the conditions for being elementary augmenting path are satisfied in k_i, too.

b) $\deg^R_W(c) = 0$, $\deg^B_W(c) = 2$
Using the notation and the observations of the previous case it is easy to see, that we can take any of the two arcs of C connecting k_i to k_j.

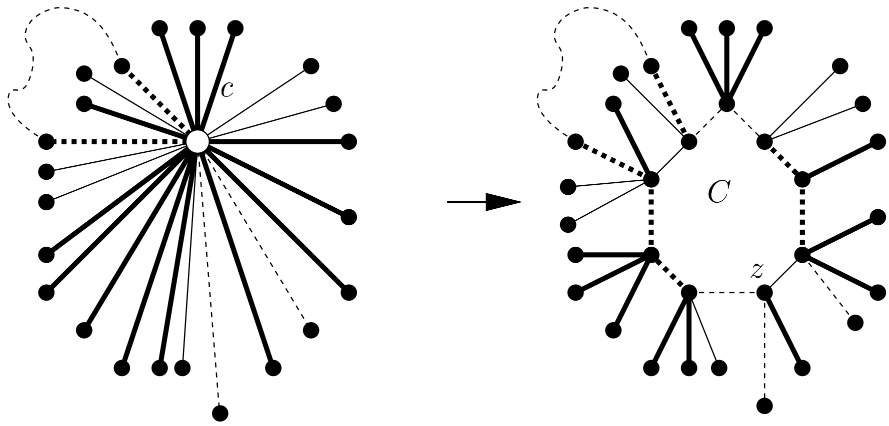

Figure 4: Blowing up c to C.

c) $\deg_W^R(c) = 2$, $\deg_W^B(c) = 2$
Let k_i and k_j be the vertices of the cycle which are the endvertices of the red edges, and k_n and k_m the ones of the blue edges. Since W' is elementary and $\deg_W^R(c) = 2$ one of the red edges connects c to its parent. This edge in the original graph must have the same property, so w. l. o. g. we may assume that $k_i = z$. Also it may be assumed that they are four different vertices. If not, the proof is similar.

If at least one of the edges of C incident to k_j is blue, then take that arc of C connecting k_j to the next vertex on the cycle among z, k_n and k_m which starts with this blue edge. Then connect the remaining two vertices with the arc between them. (See Fig. 4.) Using the observations of case a) it can be proved that the resulting W is an elementary augmenting-walk, only two part that needs more attention.

Since $\deg_W^R(c) = 2$, we have $s(c) = UNSAT$, but it would be possible that $s(z) = SAT$ and $\deg_W^R(z) = 2$, violating the conditions for the augmenting-walk. However, by the algorithm, the only reason to set $s(c) = UNSAT$ is that the two edges of the cycle incident to z are blue, so $\deg_W^R(z) = 2$ is impossible.

The other problem is that it might happen that we connect k_i, k_j, k_n, k_m in a wrong way: we obtain a separate augmenting-walk and a cycle (or cyclic walk). In this case we only keep the augmenting-walk part, and forget about the cycle.

d) $\deg_W^R(c) = 2$, $\deg_W^B(c) = 0$
This case can be handled with similar methods to the above cases.

Case ii) *c is not the root of G', and the length of C is even.*

The argument is similar to the above. The dummy edge added to c is only for preserving parity. It is never used in the augmenting-walk, so the proof is the same as in Case i).

Case iii) *c is the root of G'.*

In this case the difference is that W' starts in c, but it is easy to construct W using the above methods. $\square$

We are now prepared to give the algorithm of building an F-alternating tree. Fix an F-exposed vertex r and start to build an F-alternating tree from it by using the extension an shrinking steps. If augmentation is possible then enlarge F, choose a new F-exposed vertex r.

Build an F-alternating tree T.

Input: A $(1, f)$-odd subgraph F of H and an F-exposed vertex r.

Action:

While $\exists\, vw \in E(G)$ with $v \in B(T), w \notin A(T)$ do {
Case: $w \notin V(T)$, w is F-exposed
Use vw to construct an elementary F–augmenting-walk;
Extend this into an elementary F–augmenting-walk in H using Lemma 11 and use it to augment F to obtain F' in H;
Replace F by F', G by H;
If $\nexists$ an F-exposed vertex then ***Stop***
else replace T by $(\{r\}, \emptyset)$ where r is F-exposed;
Case: $w \notin V(T)$, w is F-covered
Use vw to extend T;
Case: $w \in B(T)$
Use vw to shrink a cycle, update G, F and T;
}

When the above procedure terminates then there is no vw edge with $v \in B(T), w \notin A(T)$. Then we can prove that a maximum $(1, f)$-odd subgraph cannot cover all vertices of T.

Lemma 12 *For the F-alternating tree T*

$$\text{odd}(T - A(T)) \geq 1 + \sum_{v \in A(T)} f(v)$$

holds.

PROOF: First note that every red edge of T has one end in $A(T)$, by the construction of T, hence all red edges are deleted in $T - A(T)$. Thus we can restrict our argument to the blue edges of T.

Next we prove that all children of all vertices in $A(T)$ are in a different odd component of $T - A(T)$. By the definition of $A(T)$, it is clear that no child of $v \in A(T)$ is in $A(T)$, in other words $A(T)$ is a stable set, since red edges of T are neglected. It is clear that the only path between two distinct vertices v and u of T must contain either the parent of v or the parent of u. If v and u are both children of some vertices in $A(T)$, then this means that $T - A(T)$ does not contain a path between u and v. The blue degree of all vertices of T except the root r is odd, so if we delete all the vertices of $A(T)$, which are the roots of the blue subtrees of T, then the blue degree of the children of vertices in $A(T)$ decreases by one, but the blue degree of the other vertices does not change. So if we consider u which is a children of $v \in A(T)$, then the component containing u will contain several vertices with odd blue degree and one vertex, u, with even blue degree. Thus, by parity reasons, the number of vertices in the component of u must be odd.

By the same argument and by the definition of $A(T)$ it can be shown that r is in a different component from the above ones and that this component is also odd.

It remains to show that the number of children of a vertex $v \in A(T)$ is $f(v)$. This follows again from the definition of $A(T)$, since it is the set of saturated roots of blue trees. Therefore, the claim of the lemma follows from counting the total number of components containing a children of a vertex in $A(T)$ and the additional component containing r. □

When we have concluded that r cannot be covered by a maximum $(1, f)$-odd subgraph, then remove all vertices and edges of the F-alternating tree from G. Fix a new F-exposed vertex r in the remaining graph and repeat the above process until there are F-exposed vertices. A more formal description of the algorithm is the following.

Algorithm to find a maximum $(1, f)$-odd subgraph

Input: H and a function $f : V(H) \mapsto \{1, 3, 5, \ldots\}$

Action:

Set $G = H$ and $F = F^* = A^* = U^* = \emptyset$.
While $\exists$ F-exposed vertex ***do*** {
Choose an F-exposed vertex r of G and put $T = (\{r\}, \emptyset)$;
Build an F-alternating tree from r
Extend F to a $(1, f)$-odd subgraph F' of H using Lemma 11,
replace F^* by $F^* \cup (F' \cap T)$, A^* by $A^* \cup A(T)$
and U^* by $U^* \cup \{r\}$.
Remove all vertices and edges of T from G, set $F = \emptyset$.
}
Replace F^* by $F^* \cup F$.
Return F^*, a maximum $(1, f)$-odd subgraph,
U^* the set of uncovered vertices and
A^* which proves that (3) holds, so F^* is maximum.

Theorem 13 *The above algorithm terminates in $O(|V(G)|^3)$ steps, and it gives a smallest maximum $(1, f)$-odd subgraph.*

Proof: It is clear that each augmentation step decreases the number of F-exposed vertices, so there will be $O(|V(G)|)$ augmentation steps. Between augmentations, each shrinking step decreases the number of vertices in G' while changing the number of vertices not in T, and each extension step decreases the number of vertices not in T while not changing the number of vertices in G'. Hence the total number of these steps between augmentations is $O(|V(G)|)$. On the other hand, it is easy to see, that between two augmentation step every edge of the graph is scanned at most once. Hence, we may conclude, that total number of steps is $(|V(G)|^3)$.

It was shown above that F remains a smallest $(1, f)$-odd subgraph throughout, so it remains to prove that F^* is maximal. Each time the tree T is removed from the graph the only uncovered vertex, which is removed, is the root of the tree r, all other removed vertices are covered by F. So, if the while loop is executed i times, then F^* covers $|V(G)| - i$ vertices of the graph. On the other hand we can apply Lemma 12 for each removed T. It is easy to see, that there are no edges in $G - A^*$ between different T trees, hence

$$\mathrm{odd}(G - A^*) = \sum_{\forall T} \mathrm{odd}(T - A(T)) \geq \sum_{\forall T} \left(1 + \sum_{v \in A(T)} f(v) \right) = i + \sum_{v \in A^*} f(v)$$

holds. By Theorem 3 this proves that F^* is maximum. $\square$

Remark. It is not too difficult to show, that the above algorithm gives the sets $A(G), C(G), D(G)$ of Theorem 6, as well. Namely, $A(G) = A^*$, odd components of $G - A(G)$ form $D(G)$ and even components of $G - A(G)$ form $C(G)$.

References

[1] W. J. Cook, W. H. Cunningham, W. R. Pulleyblank, and A. Schrijver. *Combinatorial optimization.* John Wiley & Sons Inc., New York, 1998.

[2] G. Cornuéjols. General factors of graphs. *J. Combin. Theory Ser. B*, 45(2):185–198, 1988.

[3] Y. Cui and M. Kano. Some results on odd factors of graphs. *J. Graph Theory*, 12(3):327–333, 1988.

[4] G. Y. Katona and M. Kano. Odd subgraphs and matchings. *Discrete Math.*, 250:265–272, 2002.

[5] L. Lovász. The factorization of graphs. II. *Acta Math. Acad. Sci. Hungar.*, 23:223–246, 1972.

[6] L. Lovász and M. D. Plummer. *Matching theory.* North-Holland Publishing Co., Amsterdam, 1986.

[7] J. Topp and P. D. Vestergaard. Odd factors of a graph. *Graphs Combin.*, 9(4):371–381, 1993.