PATH FACTORS OF A GRAPH

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ABSTRACT. Our purpose is to propose a new viewpoint for graph factors, apart from the traditional degree conditions. A spanning subgraph F is called a path-factor if each component of F is a path of order at least two. In particular, a path-factor F is called a (P_2,P_3)-factor if each component of F is either P_2 or P_3. A P_n-factor F, for some fixed $n \geqq 2$, is a factor such that every component of F is a path P_n of order n. Several results on (P_2,P_3)-factors, P_3-factors and P_4-factors and their applications for the "triominos tiling problem" are presented and also some graph decomposition problems related to these factors are discussed.

1. INTRODUCTION

We deal with only finite, simple graphs, which have neither multiple edges nor loops. All notation and definitions not given here can be found in Harary [13].

Let $G_1, G_2, \ldots, G_n$ be nontrival graphs. A graph G has a $(G_1, G_2, \ldots, G_n)$-subgraph H if H is a subgraph of G and each component of H is isomorphic to one of the G_i $(i = 1,2,\ldots,n)$. In particular, such a spanning subgraph of G is called a $(G_1, G_2, \ldots, G_n)$-factor of G.

A factor F is called a path-factor (or a cycle-factor) if every component of F is a path (or a cycle). From this point of view, ordinary 1-factors (or 2-factors) are just the same as P_2-factors (or cycle-factors). Those factors defined in this manner are called component factors of a graph. Several component-factors concerning path or cycle are listed in the followings.

A List of Component-Factors related to Paths, Cycles or Stars

1. P_2-factor; 1-factor, see Tutte [16].
2. P_3-factor
3. In general, P_i-factor for some fixed $i \geqq 2$.
4. (P_2, P_3)-factor; (1,2)-factor, see Akiyama, Avis and Era [3] or Akiyama [1,2].
5. $(P_2, C_n | n \geqq 3)$-factor, see Tutte [17], Hajnal [12] and Berge [9, Theorem 3.1].
6. $(P_2, C_{2n+1} | n > 1)$-factor, see Cornuejols and Pullyblank [10].
7. (P_2, C_3)-factor
8. cycle-factor; 2-factor, see Belck [7].
9. star-factor, see Amahashi and Kano [6].

2. (P_2, P_3)-FACTORS

A spanning subgraph F of G is called a (1,2)-factor if each vertex of F has degree 1 or 2 in F. A criterion for a graph to have a (1,2)-factor was discovered by Akiyama, Avis and Era [3]. The following three statements are equivalent.

(a) G has a (1,2)-factor

(b) G has a path-factor

(c) G has a (P_2, P_3)-factor

Hence, the criterion for (a) is the same as for (b) or (c):

Theorem A. A graph G has a (P_2, P_3)-factor if and only if

$$i(G-S) \leqq 2|S|$$

for every subset $S \subset V(G)$, where $i(G-S)$ denotes the number of isolated vertices of $G-S$.

This theorem has some interesting corollaries as follows.

Corollary A1. Let G be a graph with maximum degree Δ and minimum degree δ. If $\Delta/\delta \leqq 2$, then G has a (P_2, P_3)-factor.

Proof. Let S be a subset of V(G). Then the inequality $i(G-S) \cdot \delta \leqq \Delta \cdot |S|$ holds, and thus $i(G-S) \leqq 2|S|$ since $\Delta/\delta \leqq 2$. $\square$

The next corollary follows at once from the previous result since $\Delta = \delta$ for regular graphs.

Corollary A2. Every regular graph has a (P_2,P_3)-factor. ▯

Every maximal planar graph has a (P_2,P_3)-factor although its ratio Δ/δ may become large.

Corollary A3. Every maximal planar graph has a (P_2,P_3)-factor.

Proof. Let G be a maximal planar graph, S be any vertex subset of G, and $I(G-S)$ be the set of isolated vertices of $G-S$. Then the neighborhood $N(G-S)$ of $I(G-S)$ is contained in S and the subgraph H induced by $N(G-S)$ of G forms a planar graph without end vertices, since G is maximal planar.

For any component C of H, we denote by $r(C)$ the number of regions of C. Then applying the Euler Polyhedron Formula, we obtain

$$r(C) \leqq 2|V(C)| - 4 \leqq 2|V(C)|.$$

Hence we have the following inequalities:

$$i(G-S) = |I(G-S)| \leqq \sum_{C\subset H} r(C)$$

$$\leqq \sum_{C} 2|V(C)| \leqq 2|V(H)| \leqq 2|S|.$$

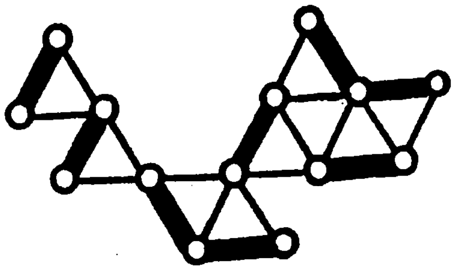
Consequently, G has a (P_2,P_3)-factor by Theorem A. ▯

We introduce a family of graphs called triangle graphs. Let π be a plane with rectangular coordinates and L be a set of lines on π given by:

$$L = \{y=n \quad \text{or} \quad y=\pm\sqrt{3}\,x+2n \quad | n \in Z\}.$$

An infinite graph I is obtained by taking the set of lattice points of π as the vertex set $V(I)$ and the set of unit segments of π as the edge set $E(I)$.

A <u>triangle graph</u> is a subgraph of I obtained from a set of unit triangles on π, see Figure 1.



<u>Figure 1</u>. A triangle graph T and its (P_2, P_3)-factor

Corollary A5. Every triangle graph T has a (P_2, P_3)-factor.

<u>Proof</u>. Let S be a vertex subset of T. It is easy to see that for every vertex $x \in S$, the number of edges joining x and the isolated vertices of $G - S$ does not exceed 4. Hence we have the following inequalities.

$$2i(G - S) \leqq 4|S|. \qquad \square$$

3. P_4-FACTORS

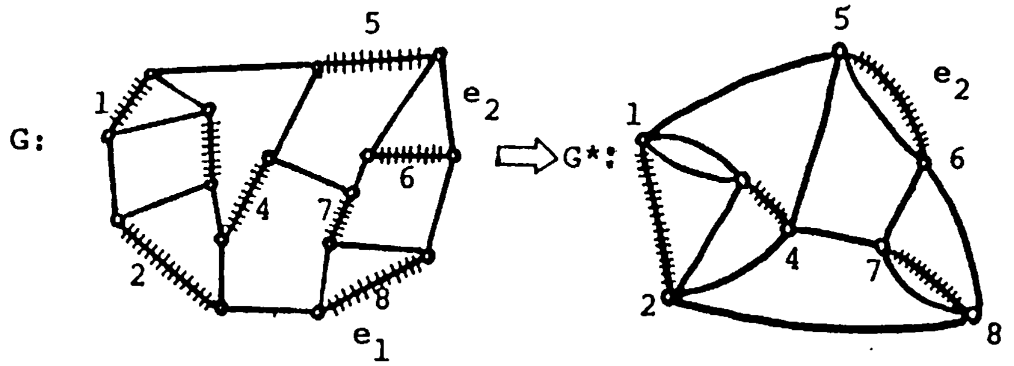
We present the following theorem which is analogous to the theorem of Petersen which shows the existence of a P_2- factor for cubic bridgeless graphs.

The next lemma is required to provem Theorem 3.1.

Lemma B. (Berge [8, Theorems 6 and 7 in Chapter 18] and Plesnik [14]). Let G be an r-regular, $(r-1)$-edge connected multigraph of even order. Then there exists a P_2-factor containing an arbitrarily given edge e. ▯

Theorem 3.1. Let G be a 3-edge connected, cubic graph of order 4p. Then for any two edges e_1 and e_2, there exists a P_4-factor containing both of them.

<u>Proof</u>. A 3-edge connected, cubic graph G has a P_2-factor F_1 containing e_1 by Lemma B. Denote by G^* the graph obtained from G by contracting every edge of F_1. (See Figure 2.)



: a P_2-factor F_1 : a P_2-factor F_2

<u>Figure 2</u>.

Then G^* is a 4-regular, 3-edge connected multigraph. Applying Lemma B, G^* has a P_2-factor F_2, which contains e_2 if F_1 does not contain e_1.

Then $F_1 \cup F_2$ constitutes a P_4-factor of G and it contains both e_1 and e_2. ▯

Corollary 3.2. Every 3-connected cubic graph of order 4p has a P_4-factor.

The graph illustrated in Figure 3 shows that the

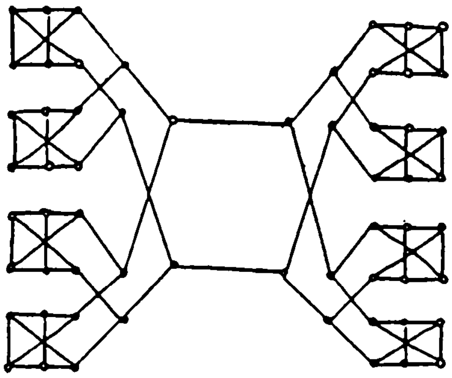


Figure 3. A 2-edge connected cubic graph of order 60 having no P_4-factor

connectivity hypothesis of Theorem 3.2 cannot be omitted

4. P_3-FACTORS

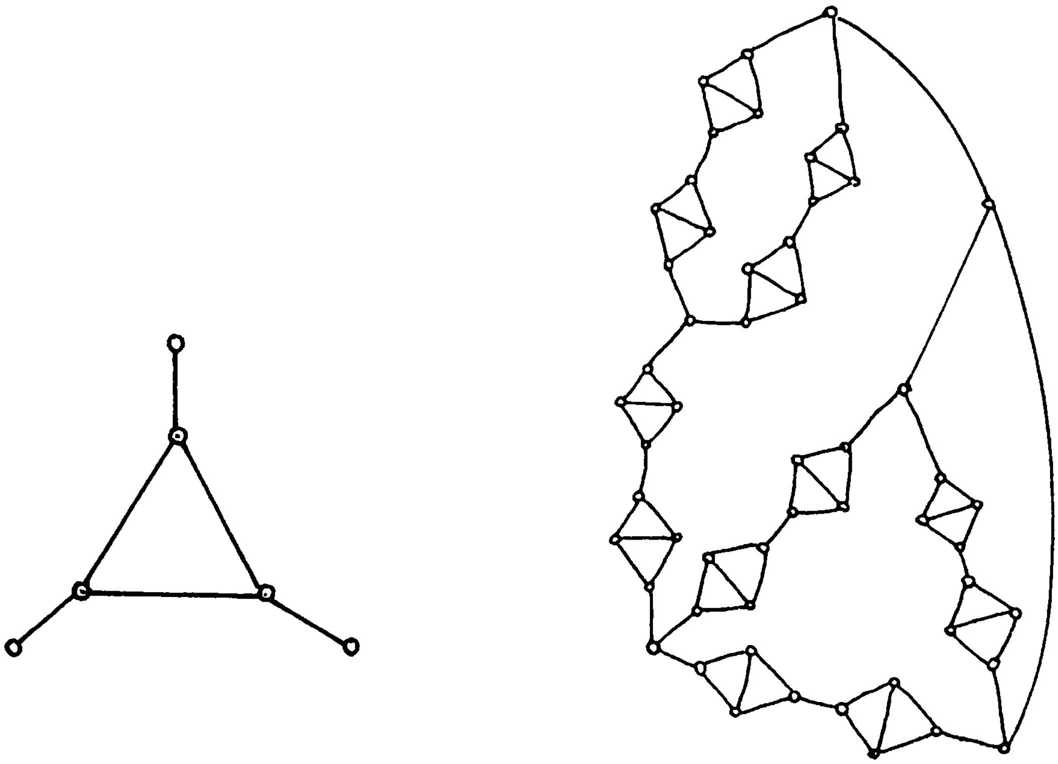
We first discuss an easy necessary condition for a graph to have a P_3-factor.

Suppose G has a P_3-factor. Then for any vertex subset S of G, the components of $G - S$ can be classified into three types T_i $(i=0,1,2)$ according as their order i (mod 3). Denote by $\omega_i(G-S)$ the number of components of the type T_i, then we have the inequality (4.1) by estimating the least number of vertices of S needed to form a P_3-factor of G.

$$\omega_1(G-S) + 2\omega_2(G-S) \leqq 2|S| \tag{4.1}$$

However the condition (4.1) is not sufficient for a graph to have a P_3-factor, which can be seen in Figure 4(a).

We here propose a conjecture on the existence of a P_3-factor for cubic graphs.



(a) A graph without P_3-factors, which satisfies (4.1)

(b) A 2-connected cubic graph of order 54 with no P_3-factors

Figure 4.

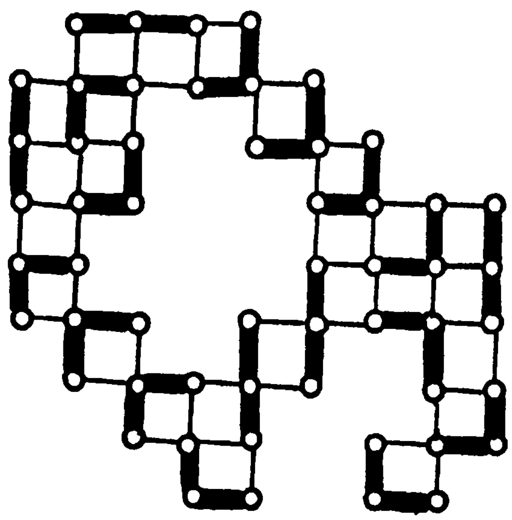
Conjecture. Every 3-connected cubic graph of order 3p has a P_3-factor.

Note that there exists a 2-connected cubic graph of order 3p with no P_3-factors as illustrated in Figure 4b.

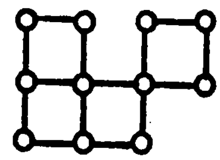
We shall introduce a special family of graphs, which is motivated from "the triominos tiling problem" in the next section.

Let π be a plane with rectangular coordinates and I be a set of the lattice points on π, that is, $I = \{(i,j) \mid$ both i and j are integers$\}$. We define a graph G on the plane π as follows:

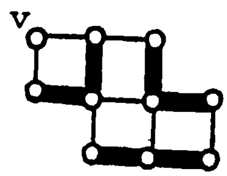
Take a finite subset $V \subset I$ as the vertex set of G and join every two vertices x, y of V if and only if the distance between x and y is 1. If a graph G obtained in this manner satisfies the property that G is connected and every edge of G is contained in some 4-cycle C_4 of G (which will be referred to as a <u>square</u> hereafter), then G is called a <u>square graph</u>, see Figure 5a. Note that the graph in Figure 5b is not a square graph.



(a) A square graph and its P_3-factor



(b) A graph which is not a square graph



(c) G - v has a P_3-factor

<u>Figure</u> 5.

Theorem 4.1. Every square graph of order 3p has a P_3-factor.

Theorem 4.1 is an immediate consequence of the next theorem, for which we shall give a brief outlint of the proof.

Theorem 4.2. Let G be a square graph of order p.

(I) If $p \equiv 0 \pmod 3$, G has a P_3-factor,(Fig. 5a)

(II) If $p \equiv 1 \pmod 3$ and v is an arbitrary vertex of degree 2 in G, G - v has a P_3-factor, (Fig. 5c)

Outline of proof. For convenience, we say a v-semifactor of G is a P_3-subgraph which contains all the vertices of G except a single vertex v. By the capital letters A, B, C,..., we denote squares (faces) on the plane, and by the small letters a,b,c,..., we denote vertices of G. Two squares (faces) A and B intersect if they have a common vertex of G, and they are adjacent if they have a common edge of G.

Our proof is by induction on the order p of G, and it is shown in Figure 6 that all square graphs of small order p with $p \equiv 0$ or 1 (mod 3) have a P_3-factor or a v-semifactor respectively, as the base of induction.

(I) *G is a square graph of order $p \equiv 0 \pmod 3$.*

Lemma 4.1. If there is a square of G which is not adjacent to any other square, then G has a P_3-factor.

Proof. For convenience, we name the squares (faces) and vertices of G as in Figure 7a.

Suppose that x is not a cutvertex of G. Then $G-\{a,b\}$ has a c-semifactor by the induction hypothesis and so G has a P_3-factor.

Suppose that x is a cutvertex of G. If $G \not\supset C$, then $G-\{a,b,c\}$ has a P_3-factor by the hypothesis, and thus G has a P_3-factor. Hence we may assume that $G \supset B,C$. Set $G-\{a,b\} = H \cup K$ such that $H \supset C$, $K \supset B$. Then both H and K are square graphs. We now divide our proof into three cases.

Case 1. $|V(H)| \equiv 0$ (or 1) (mod 3) By the hypothesis, H (or K) has a P_3-factor and K (or H) has a x-semifactor (or c-semifactor), respectively. Therefore, G has a P_3-factor.

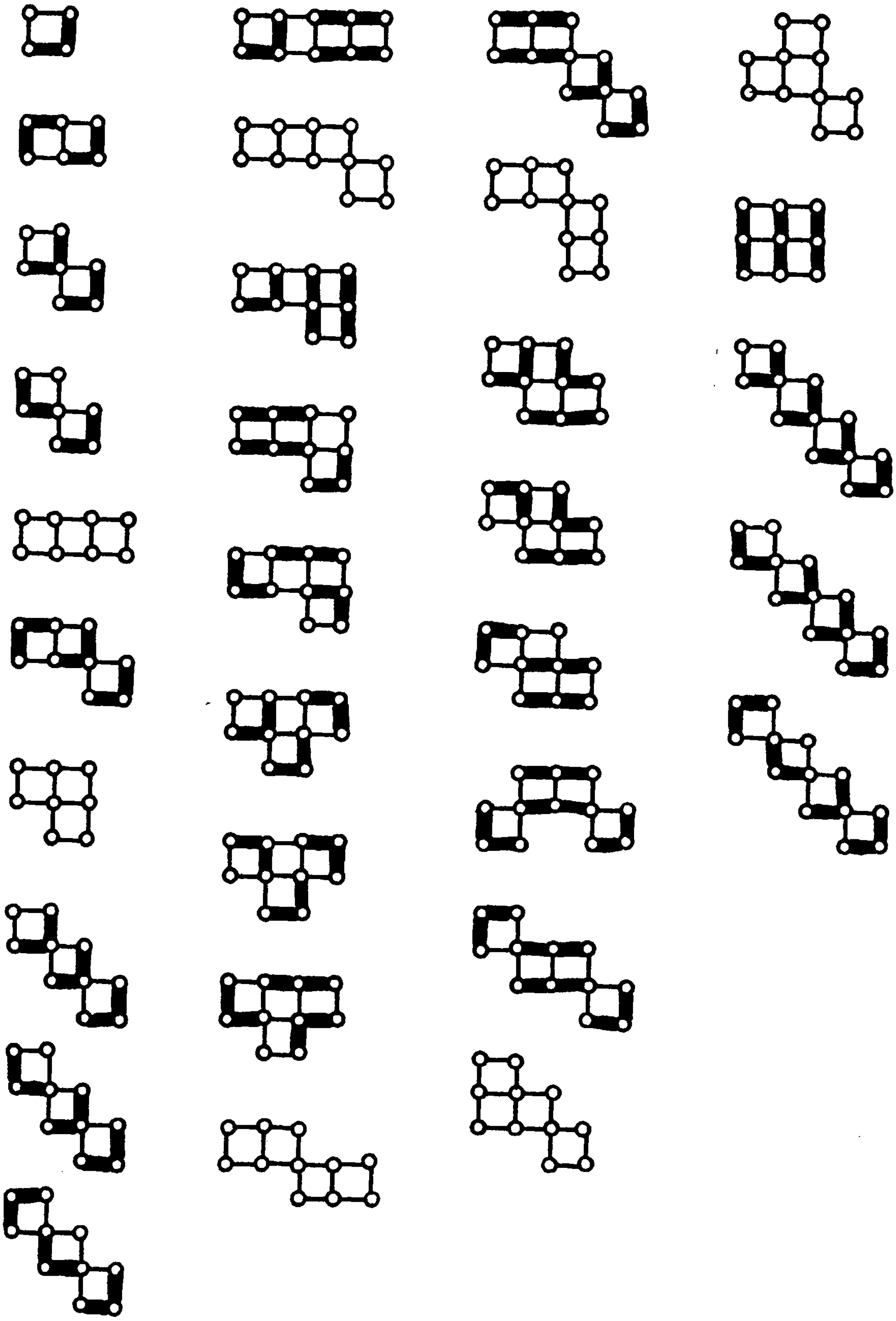
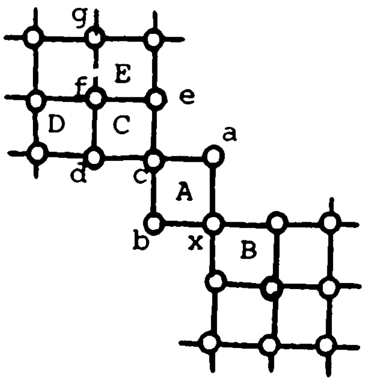
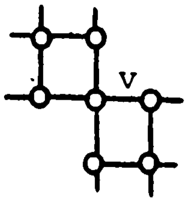


Figure 6. Small square graphs and their P_3-factors or v-semifactor when $p \equiv 0$ or 1, respectively



(a)



(b)

Figure 7.

Case 2. $|V(H)| \equiv 2 \pmod 3$ If $G \supset D$ and $G \not\supset E$, then $H-\{c,e\}$ has a P_3-factor. If $G \supset D$ and E. then $H - c$ has e-semifactor. If $G \not\supset D$, E, then $G \supset F$. In this case, it can be shown inductively that $H-\{c,d,e,f,g\}$ has a P_3-factor by inspection. Hence $H + a$ has a P_3-factor. Similarly, we see that $K+b$ has a P_3-factor. Consequently, G has a P_3-factor. ▯

We require three more lemmas in order to prove (I) of Theorem 4.2, but as their proofs are long and monotonous, we shall omit them.

Lemma 4.2. If G has a vertex v which is contained in exactly two squares (see Figure 7b), then G has a P_3-factor. ▯

Lemma 4.3. If G has a square which has exactly one adjacent square, then G has a P_3-factor. ▯

Lemma 4.4. If every square of G has at least two adjacent squares, then G has a P_3-factor. ▯

(II) G is a connected square graph of order 1 (mod 3) and v is an arbitrary vertex of G with degree 2.

Orientation of Proof of (II). For convenience, we name the square of a part of G as shown in Figure 8.

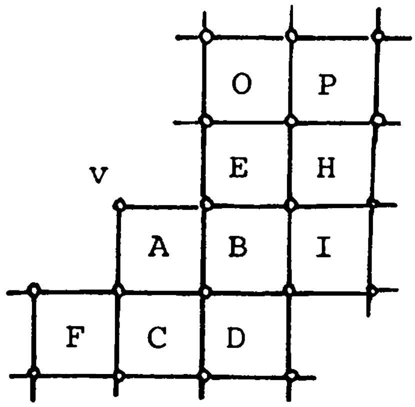


Figure 8.

If $G \supset B, C$, then $G - v$ has a P_3-factor by induction. Hence we may assume that $G \not\supset C$. Moreover, if $G \not\supset B, C$ and $G \supset D$, then $G \not\supset E, F$ which implies that G has a v-semifactor by induction.

The proof is completed by considering the following 12 cases and proving that G has a v-semifactor in each case by induction. However, since the proof of each case is long and tedious we only list the cases:

Case 1. $B, D \not\subset G$

Case 2. $B \subset G$, $D \not\subset G$, $F \subset G$, $E \subset G$

Case 3. $B \subset G$, $D \not\subset G$, $F \subset G$, $E \not\subset G$

Case 4. $B \subset G$, $D \not\subset G$, $F \not\subset G$, $E \subset G$

Case 5. $B \subset G$, $D \not\subset G$, $F \not\subset G$, $E \not\subset G$

Case 6. $B \subset G$, $D \subset G$, $E \not\subset G$,

Case 7 $B \subset G$, $D \subset G$, $E \subset G$, $O \subset G$, $I \subset G$

Case 8. $B \subset G$, $D \subset G$, $E \subset G$, $O \subset G$, $I \not\subset G$

Case 9. $B \subset G$, $D \subset G$, $E \subset G$, $O \not\subset G$, $P \subset G$, $I \subset G$

Case 10. $B \subset G$, $D \subset G$, $E \subset G$, $O \not\subset G$, $P \subset G$, $I \not\subset G$

Case 11. $B \subset G$, $D \subset G$, $E \subset G$, $O \not\subset G$, $P \not\subset G$, $H \subset G$.

Case 12. $B \subset G$, $D \subset G$, $E \subset G$, $O \not\subset G$, $P \not\subset G$, $H \not\subset G$. $\square$

5. APPLICATIONS OF P_3-FACTORS FOR TRIOMINOS TILING PROBLEMS

We remove an arbitrary number of squares from an $m \times n$ chessboard so that the remaining part is connected, and call it a defective board or more briefly a d-board. Two unit squares of a defective chessboard are adjacent if they have a common edge. A d-board B is said to be tough if every pair of adjacent unit squares is contained in a 2×2 subsquare of B. A tough d-board is illustrated in Figure 9.

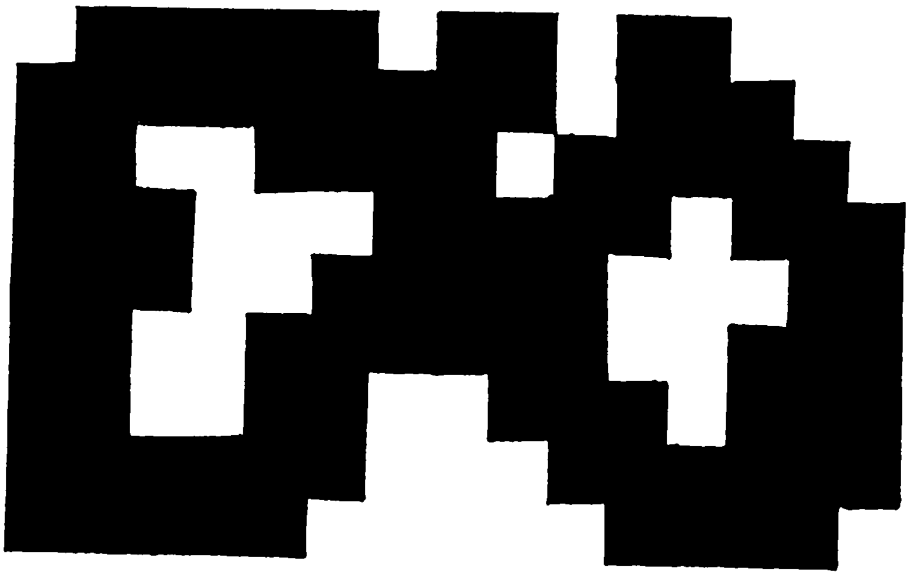
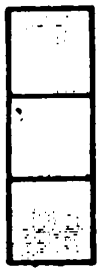
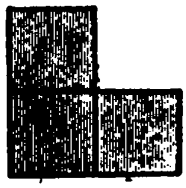


Figure 9. A tough d-board (black part)

There are exactly two kinds of triominos which have different shape called Tic (Figure 10a) and El (Figure 10b).



(a)



(b)

Figure 10. Tic and El

By tiling a d-board B with triominos we mean covering each square of B exactly once without parts of the triominos extending over the removed square or the edges of the board (see examples in Figure 11).

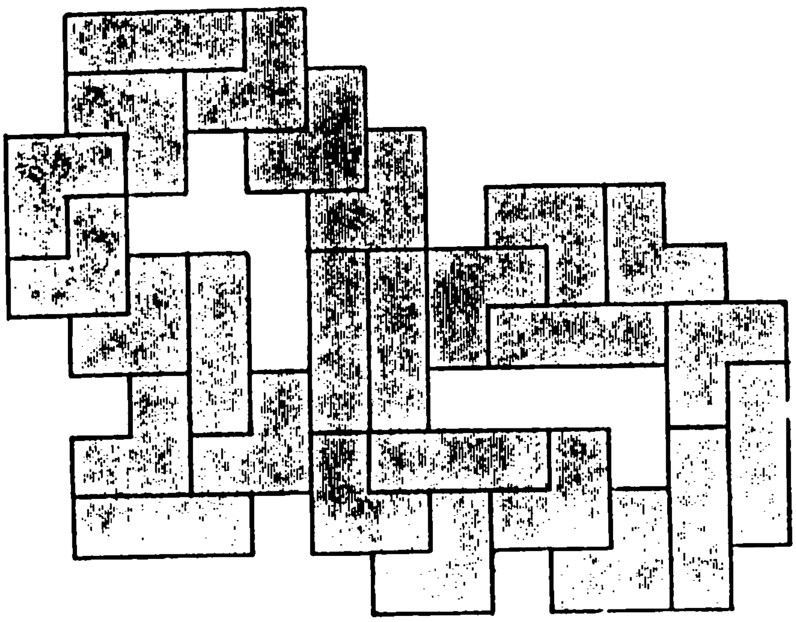


Figure 11. Tiling a tough d-board by triominos

A square A of a d-board B is called a <u>corner square</u> if A is adjacent with exactly two squares of B.

The <u>order</u> of a d-board B is the number of unit squares of B.

Theorem 5.1. (I) Every tough defective chessboard of order 0 (mod 3) can be tiled with triominos.
(II) Every tough d-board of order 1 (mod 3) can be tiled with triominos except an arbitrary prescribed corner square.

<u>Proof</u>. For a given tough d-board B, we define the graph G(B) such that the vertices of G(B) represent the unit squares of B and the edges of G(B) represent the adjacency of the two corresponding unit squares of B and call those graphs <u>tough graphs</u>. The graph G(B) corresponding to the tough d-board in Figure 11 is illustrated in Figure 12.

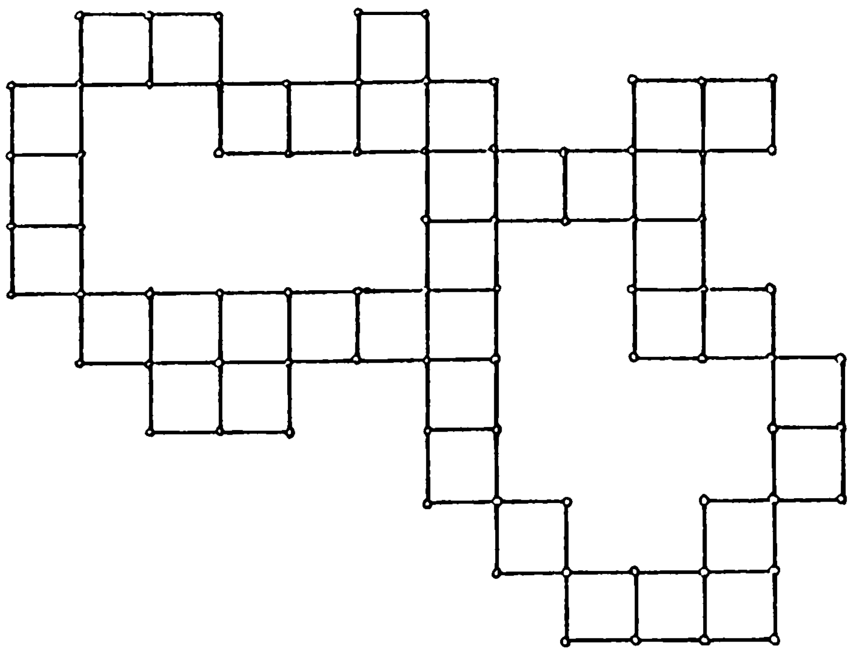


Figure 12. The tough graph corresponding to the tough d-board of Figure 11

It is easily verified that every tough graph is a square graph as defined in Section 5. Applying Theorem 4.2 to G(B), we see that G(B) has a P_3-factor, which implies the theorem. ▯

6. LINEAR ARBORICITY AND STAR DECOMPOSITION INDEX

Let $(G_1, G_2, \ldots, G_n)$ be a set of graphs. If a graph G can be partitioned into edge-disjoing union of $(G_1, G_2, \ldots, G_n)$-subgraphs of G, then the minimum number of those subgraphs is called the $(G_1, G_2, \ldots, G_n)$-<u>subgraph Decomposition Index of G</u>. In particular, the <u>star decomposition index</u> of G, denoted by $*(G)$, has a star for each G_i, and the $(P_n | n \geq 2)$-subgraph decomposition index is the same as the linear arboricity, $\Xi(G)$.

The linear arboricity for any r-regular graph G was conjectured to be $\lceil (r+1)/2 \rceil$ in [2]. This has been proved when r = 3,4 in Akiyama, Exoo and Harary [4,5], r = 6 in Tomasta [15], r = 5,6 in Enomoto and Peroche [11].

We now present a few uses of the (P_2, P_3)-factor theorem in linear arboricity problems by showing much shorter proofs than the original ones.

Theorem 6.1. Every cubic graph has linear arboricty 2.

<u>Proof</u>. By Corollary A2, there exist path-factors F of G. Let F' be a path-factor of G having the maximum size among all F's. Denote by H the graph obtained by deleting all edges of F' from G. We claim that H is also a path-factor of G. Suppose that H contains a cycle C, then there are three consecutive vertices

v_1, v_2 and v_3 on C. One of the edges v_1v_2 or v_2v_3 could be added to F' so that either F' + v_1v_2 or F' + v_2v_3 is still a path-factor of G, which contradicts the maximality of F'. ▯

Theorem 6.2. Every 4-regular graph has linear arboricity 3.

Proof. By Corollary A2, G has a path-factor F. Denote by H the graph obtained from G by deleting all edges of F. Since H can be embedded in some cubic graph, it is union of path-subgraphs F_1 and F_2 by Theorem 6.1. Therefore G is the union of F, F_1, and F_2.

We now turn our attention to star decomposition index.

Theorem 6.3. Let $n \geqq 4$. Then the star decomposition index of the complete graph of order n is $\lceil n/2 \rceil + 1$, i.e.,

$$*(K_n) = \lceil n/2 \rceil + 1$$

Proof. We first show the lower bound $*(K_n) \geqq \lceil n/2 \rceil + 1$ by induction on order n. It suffices to prove that $*(K_{2m-1}) \geqq m+1$ since $*(K_{2m}) \geqq *(K_{2m-1})$ and $\lceil 2m/2 \rceil = \lceil (2m-1)/2 \rceil = m$.

Suppose that $K_{2m-1} = F_1 \cup F_2 \cup \dots \cup F_k$, where each F_i is a star subgraph of K_{2m-1}. If some F_i is a $K(1,2m-2)$, then $K_{2m-1} - F_i = K_{2m-2} \cup K_1 = F_1 \cup \dots \cup F_{i-1} \cup F_{i+1} \cup \dots \cup F_k$ and so $k-1 \geqq *(K_{2m-2}) \geqq m$ by the induction hypothesis. Hence $k \geqq m+1$. Therefore, we may assume that $F_i \not\cong K(i,2m-2)$ for every i and thus $|E(F_i)| \leqq 2m-3$. Then $|F_1 \cup \dots \cup F_k| \leqq k(2m-3)$. On the other hand, $|E(K_{2m-1})| = (2m-1)m$. Hence we obtain $k \geqq m+1$.

We next show that $*(K_n) \leqq \lceil n/2 \rceil + 1$. It suffices to prove that $*(K_{2m}) \leqq m+1$ since $*(K_{2m-1}) \leqq *(K_{2m})$ and $\lceil 2m/2 \rceil = \lceil (2m-1)/2 \rceil = m$. Let $V(K_{2m}) = \{v_i \mid i=1,2,\ldots, 2m\}$ and put

$$F_\ell = \{v_\ell v_i \mid \ell+1 \leqq i' < \ell+m,\ i \equiv i' \pmod{2m}$$

$$\cup \{v_{m+\ell} v_j \mid m+\ell+1 \leqq j' < 2m+\ell,\ j \equiv j' \pmod{2m}\}$$

for $\ell = 1,2,\ldots,m$,

and

$$F_{m+1} = \{v_1 v_{m+1},\ v_2 v_{m+2}, \ldots, v_m v_{m+m}\}.$$

Then

$K_{2m} = F_1 \cup F_2 \cup \cdots \cup F_{m+1}$, and thus we have $*(K_{2m}) \leqq m+1$. $\square$

Acknowledgement

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