Disjoint odd integer subsets having a constant even sum

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Abstract

We prove that for position integers n and k and a positive even integer m, the odd integer set $\{1, 3, 5, \ldots, 2n-1\}$ contains k disjoint subsets having a constant sum m if and only if $4k \leq m \leq n^2/k$, $n^2 - mk \neq 2$ and either $m \neq 4n-2$ or $n \neq 4k$.

1 Introduction

The following theorem was obtained by Ando et al.[2].

Theorem A. *Let n, m and k be positive integers. Then the integer set $\{1, 2, \ldots, n\}$ contains k disjoint subsets $A_1, A_2, \ldots, A_k$ such that the sum of all elements of each A_i is equal to m if and only if $2k - 1 \leq m \leq \frac{n(n+1)}{2k}$.*

The following conjecture is also given by [2].

Conjecture. *Let n, m and k be positive integers. Then the odd integer set $\{1, 3, \ldots, 2n-1\}$ contains k disjoint subsets having a constant sum m if and only if one of the following two conditions holds:*

(ca) m is even, $4k \leq m \leq n^2/k$, $n^2 - mk \neq 2$, and either $n \neq 4k$ or $m \neq 16k - 2$.

(cb) m is odd, and either $9(k-1) \leq m \leq 2n-1$, or $9k \leq m \leq n^2/k$ and $n^2 - mk \neq 2$.

In this paper we shall prove Conjecture(ca). First, we give an example. Let $n = 12$, $m = 38$ and $k = 3$. Then n, m and k satisfy the conditions of (ca), and the set $\{1, 3, 5, \ldots, 23\}$ contains three disjoint subsets $A_1 =$

$\{1, 7, 9, 21\}$, $A_2 = \{3, 5, 11, 19\}$ and $A_3 = \{15, 23\}$ with sum 38. Note that Fu and Hu[4] has recently proved Conjecture(cb).

We can consider the following general problem. Let A be a finite set of integers, and $m_0, m_1, \ldots, m_k$ be positive integers such that

$$\sum_{x \in A} x = m_0 + m_1 + \cdots + m_k.$$

Under what conditions can we partition A into disjoint subsets $A_0, A_1, \ldots, A_k$ such that

$$\sum_{x \in A_i} x = m_i \quad \textit{for all } i, \ 0 \le i \le k?$$

In Theorem A, $A = \{1, 2, \ldots, n\}, m_1 = \cdots = m_k = m$ and $m_0 = n(n+1)/2 - (m_1 + \cdots + m_k)$. In our present theorem, we deal with the case that $A = \{1, 3, \ldots, 2n-1\}$, $m_1 = m_2 = \cdots = m_k = m$ and $m_0 = n^2 - (m_1 + \cdots + m_k)$. In particular, it may be interesting to find sufficient conditions for a set $A = \{a, a+d, a+2d, \ldots, a+(n-1)d\}$ to be partitioned into disjoint subsets $A_0, A_1, \ldots, A_k$ such that $A_0, A_1, \ldots, A_k$ such that

$$\sum_{x \in A_i} x = m \quad \textit{for all } i, \ 1 \le i \le k.$$

Moreover, the following conjecture is proposed by [1] and [3].

Conjecture. *Let m_i, $0 \le i \le k$, be integers such that $n < m_0 \le m_1 \le \cdots \le m_k \le 2n-2$ and $m_0 + \cdots + m_k = n(n+1)/2$. Then $\{1, 2, \ldots, n\}$ can be partitioned into $k+1$ subsets with sum m_i, $0 \le i \le k$.*

2 The result

In this section, we shall prove the following stronger theorem, which contains Conjecture(ca) as a corollary.

Theorem. *Let n and k be positive integers and m be a positive even integer. Then the following two statements holds:*
(a) $\{1, 3, \ldots, 2n-1\}$ contains k disjoint subsets with sum m if and only if (i) $4k \le m \le n^2/k$; (ii) $n^2 - mk \ne 2$; and(iii) $n \ne 4k$ or $m \ne 16k - 2$.
(b) $\{1, 3, \ldots, 2n-1\}$ contains $k+1$ disjoint subsets $A_1, \ldots, A_k$ each with sum m and B with sum $m/2$ if and only if (iv)either $m \equiv 0 (mod 4)$ and $4k+8 \le m \le 2n^2/(2k+1)$ or $m \equiv 2 (mod 4)$ and $4k+2 \le m \le 2n^2/(2k+1)$; (v) $n^2 - (2k+1)m/2 \ne 2$; and (vi) $n \ne 4k+3$ or $m \ne 16k+14$.

Proof. We first prove the necessity of statement (a). Suppose that $\{1, 3, \ldots, 2n-1\}$ contains k disjoint subsets $A_1, A_2, \ldots, A_k$ having the constant sum m. Since m is even, each A_i contains at least two elements, and so $|A_1 \cup \cdots \cup A_k| \geq 2k$. Thus we have

$$mk = \sum_{i=1}^{k} \sum_{x \in A_i} x \geq 1 + 3 + \cdots + (4k-1) = 4k^2.$$

Therefore, $m \geq 4k$. On the other hand, we have $mk \leq 1+3+\cdots+(2n-1) = n^2$, and so $m \leq n^2/k$.

Assume $n^2 - mk = 2$. Then

$$n^2 - mk = 1 + 3 + \cdots + (2n-1) - \sum_{i=1}^{k} \sum_{x \in A_i} x = 2.$$

This is impossible. Hence, if $n^2 - mk = 2$, then $\{1, 3, \ldots, 2n-1\}$ does not contain k disjoint subsets with sum m.

We next assume that $n = 4k$ and $m = 16k-2$. Since $(2n-1)+(2n-3) < m$, each A_i contains at least four elements, and so $|A_1 \cup \cdots \cup A_k| \geq 4k$. Thus

$$mk = \sum_{i=1}^{k} \sum_{x \in A_i} x \geq 1 + 3 + \cdots + (8k-1) = 16k^2.$$

Hence $m \geq 16k$. This contradicts $m = 16k-2$. Therefore, if $n = 4k$ and $m = 16k-2$, then we cannot obtain k disjoint subsets with sum m from $\{1, 3, \ldots, 2n-1\}$.

We next prove the necessity of (b). Suppose that $\{1, 3, \ldots, 2n-1\}$ contains $k+1$ disjoint subsets $A_1, A_2, \ldots, A_k$ each with sum m and B with sum $m/2$. Since

$$(2k+1)\frac{m}{2} = \sum_{i=1}^{k} \sum_{x \in A_i} x + \sum_{x \in B} x \leq 1 + 3 + \cdots + (2n-1) = n^2,$$

we have $m \leq 2n^2/(2k+1)$. If $m \equiv 0 (mod 4)$, then $|A_1 \cup \cdots \cup A_k \cup B| \geq 2k+2$ as $m/2$ is even, and so $(2k+1)m/2 \geq 1 + 3 + \cdots + (4k+3) = 4(k+1)^2$. Thus $m \geq 4k+8$. If $m \equiv 2 (mod 4)$, then $|A_1 \cup \cdots \cup A_k \cup B| \geq 2k+1$ and so $(2k+1)m/2 \geq (2k+1)^2$. Hence $m \geq 4k+2$. Suppose that $n = 4k+3$ and $m = 16k+14$. Since $m > (2n-3)+(2n-1)$ and $m/2 > 2n-1$, we have $|A_1 \cup \cdots \cup A_k \cup B| \geq 4k+3$. Thus $(2k+1)m/2 \geq (4k+3)^2 = n^2$. On the other hand, we have $(2k+1)m/2 = (2k+1)(8k+7) < n^2$, which is a contradiction.

We now prove the sufficiency by induction on n. It is easy to see that the theorem holds if $n \leq 5$. Hence we may assume $n \geq 6$. Since $4k \leq m \leq n^2/k$ or $4k+2 \leq m \leq 2n^2/(2k+1)$, we have $2k \leq n$ or $2k+1 \leq n$. We consider four case.

Case1. (a) with $m \geq 4n$. Let $B_1 = \{2n-1, 2n-4k+1\}$, $B_2 = \{2n-3, 2n-4k+3\}, \ldots, B_k = \{2n-2k+1, 2n-2k-1\}$, and let $n^{'} = n-2k$ and $m^{'} = m-4n+4k$. Then $4k \leq m^{'} \leq n^{'2}/k$ and $n^{'2} - m^{'}k \neq 2$ since $m \geq 4n$ and $n^2 - mk = n^{'2} - m^{'}k \geq 0$. If $n^{'} = 4k$ and $m^{'} = 16k-2$, then $n = 6k$ and $m = 36k-2$. Hence, if $n \neq 6k$ or $m \neq 36k-2$, then $n^{'}, m^{'}$ and k satisfy (i)-(iii), and so $\{1, 3, \ldots, 2n^{'}-1\}$ contains k disjoint subsets $C_1, \ldots, C_k$ with sum $m^{'}$ by the induction hypothesis. Therefore, we obtain k disjoint subsets $B_1 \cup C_1, \ldots, B_k \cup C_k$ of $\{1, 3, \ldots, 2n-1\}$ with sum m. If $n = 6k$ and $m = 3k-2$, then the following k disjoint subsets have the constant sum m:

$$
\begin{aligned}
A_1 &= \{3, 4k-5, 6k-3, 6k+1, 8k+3, 12k-1\},\\
A_2 &= \{5, 4k-7, 6k-5, 6k+3, 8k+5, 12k-3\},\\
&\vdots\\
A_{k-3} &= \{2k-5, 2k+3, 4k+5, 8k-7, 10k-5, 10k+7\},\\
A_{k-2} &= \{2k-3, 2k+1, 4k+3, 8k-5, 10k-3, 10k+5\},\\
A_{k-1} &= \{8k-3, 8k-1, 10k-1, 10k+3\},\\
A_k &= \{4k-3, 4k-1, 4k+1, 8k+1, 6k-1, 10k+1\},
\end{aligned}
$$

where $A_1 \cup \cdots \cup A_k = \{1, 3, \ldots, 2n-1\} \backslash \{1, 2k-1\}$.

Case 2. (a) with $m < 4n$. If $m < 2n-1$ then we can get the desired subsets $A_1 = \{1, m-1\}, A_2 = \{3, m-3\}, \ldots, A_k = \{2k-1, m-2k+1\}$ as $4k \leq m$. Thus we may assume $2n \leq m$.

Suppose that $m \equiv 0 (mod 4)$. Let $t = n - (m/4) \geq 1$ and $A_1 = \{m-2n+1, 2n-1\}, \ldots, A_t = \{m-2n+2t-1, 2n-2t+1\}$. If $t \geq k$, then we can get k disjoint subsets with sum m, and so we may assume $t \leq k-1$(i.e. $m \geq 4n-4k+4$). Then $n^{'} = n-2t$, m and $k^{'} = k-t$ satisfy (i)-(iii) since $n^{'2} - mk^{'} = n^2 - mk \geq 0$ and $m \equiv 0 (mod 4)$. Hence $\{1, 3, \ldots, 2n^{'}-1\}$ contains $k^{'}$ disjoint subsets $A_{t+1}, \ldots, A_k$ with sum m by the induction hypothesis. Therefore, $\{1, 3, \ldots, 2n-1\}$ contains k disjoint subsets $A_1, \ldots, A_k$ with sum m.

Next suppose that $m \equiv 2 (mod 4)$. Let $t = n - (m+2)/4 \geq 0$ and $A_1 = \{m-2n+1, 2n-1\}, \ldots, A_t = \{m-2n+2t-1, 2n-2t+1\}$. Note that if $m-2n+2t+1 = 2n-2t-1 = m/2$ and $A_1 \cup \cdots \cup A_t = \{2n-2t+1, \ldots, 2n-1\} \backslash \{m/2\}$, and if $t = 0$ then $2n-1 = m/2$. Again we may assume

$t \leq k-1$(i.e.$m \geq 4n-4k+2$). Then $n^{'} = n-2t-1$, m and $k^{'} = k-t-1$ satisfy (iv) and (v) in statement (b) since $n^{'2} - (2k^{'}+1)m/2 = n^2 - mk$ and $m \geq 4k^{'}+2$. Moreover, if $n^{'} = 4k^{'}+3$ and $m = 16k^{'}+14$, then we have $m = 4n^{'}+2 = 2(m-2n)+2$, $m = 4n-2$, $t = 0$, $n = 4k$ and $m = 16k-2$, which contradict (iii). Hence, $n^{'}$, m and $k^{'}$ also satisfy (iv). Therefore,$\{1,3,\ldots,2n^{'}-1\}$ contains $k^{'}+1$ disjoint subsets $A_{t+1},\ldots,A_{k-1}$ and B with sum m and $m/2$ by the induction hypothesis. Consequently, $\{1,3,\ldots,2n-1\}$ contains k disjoint subsets $A_1,\ldots,A_{k-1}, B\cup\{m/2\}$ with sum m.

Case 3. (b) with $m \geq 4n$. Let $C_1 = \{2n-1, 2n-4k-1\}$, $C_2 = \{2n-3, 2n-4k+1\}, \ldots, C_k = \{2n-2k+1, 2n-2k-3\}$ and $E = \{2n-2k-1\}$. Then $n^{'} = n-2k-1, m^{'} = m-4n+4k+2$ and k satisfy (iv) and (v) since $n^{'2} - (2k+1)m^{'}/2 = n^2 - (2k+1)m/2$. Moreover, if $n^{'} = 4k+3$ and $m^{'} = 16k+14$, then $n = 6k+4$ and $m = 36k+28$. Hence, if $n \neq 6k+4$ or $m^{'} \neq 36k+28$, then $n^{'}, m^{'}$ and k satisfy (iv)-(vi), and thus $\{1,3,\ldots,2n^{'}-1\}$ contains $k+1$ disjoint subsets $D_1,\ldots,D_k$ and F with sum $m^{'}$ and $m^{'}/2$ by the induction hypothesis. Consequently, $\{1,3,\ldots,2n-1\}$ contains $k+1$ disjoint subsets $C_1\cup D_1,\ldots,C_k\cup D_k$ and $E\cup F$ with sum m and $\{m/2\}$. If $n = 6k+4$ and $m = 36k+28$, then the following $k+1$ disjoint subsets have sum m and $m/2$, respectively.

$$\begin{aligned}
&A_1 = \{3, 4k+3, 4k+5, 8k+5, 8k+7, 12k+5\},\\
&A_2 = \{5, 4k+1, 4k+7, 8k+3, 8k+9, 12k+3\},\\
&\vdots\\
&A_{k-1} = \{2k-1, 2k+7, 6k+1, 6k+9, 10k+3, 10k+9\},\\
&A_k = \{2k+3, 2k+5, 6k+3, 6k+5, 10k+5, 10k+7\},\\
&B = \{6k+7, 12k+7\},
\end{aligned}$$

where $A_1\cup\cdots\cup A_k\cup B = \{1,3,\ldots,2n-1\}\backslash\{1,2k+1\}$.

Case 4. (b) with $m < 4n$. If $m < 2n-1$ and $m \equiv 0(mod 4)$, then we can get the desired subsets $A_1 = \{3.m-3\},\ldots,A_k = \{2k+1, m-2k-1\}, B = \{1,(m/2)-1\}$ as $4k+8 \leq m$. If $m < 2n-1$ and $m \equiv 2(mod 4)$, then we can get the desired subsets $A_1 = \{1.m-1\},\ldots,A_k = \{2k-1, m-2k+1\}, B = \{m/2\}$ as $4k+2 \leq m$. Hence we may assume that $2n \leq m$.

Suppose that $m \equiv 0(mod 4)$. Let $t = n-(m/4) \geq 1$ and $A_1 = \{m-2n+1, 2n-1\},\ldots,A_t = \{m-2n+2t-1, 2n-2t+1\}$. Then $n^{'} = n-2t, m$ and $k^{'} = k-t$ satisfy (iv)-(vi) since $n^{'2} - (2k^{'}+1)m/2 = n^2 - (2k+1)m/2$ and $m \equiv 0(mod 4)$. Hence $\{1,3,\ldots,2n^{'}-1\}$ contains $k^{'}+1$ disjoint subsets $A_{t+1},\ldots,A_k$ and B with sum m and $m/2$. Therefore, $\{1,3,\ldots,2n-1\}$ contains $k+1$ disjoint subsets with sum m and $m/2$, respectively.

Next suppose that $m \equiv 2(mod4)$. Let $t = n - (m+2)/4 \geq 0$ and $A_1 = \{m-2n+1, 2n-1\}, \ldots, A_t = \{m-2n+2t-1, 2n-2t+1\}$ and $B = \{2n-2t-1\} = \{m/2\}$. Then $A_1 \cup \cdots \cup A_t \cup B = \{m-2n+1, \ldots, 2n-1\}$, and $n^{'} = n-2t-1$, m and $k^{'} = k-t$ satisfy (i) and (ii) in statement (a) by the fact that $n^{'2} - k^{'}m = n^2 - (2k+1)m/2$. If $n^{'} = 4k^{'}$ and $m = 16k^{'} - 2$, then $m = 4n+2$, which contradicts the assumption of this case. Hence $n^{'}$, m and $k^{'}$ satisfy (iii). Therefore, $\{1, 3, \ldots, 2n^{'}-1\}$ contains $k^{'}$ disjoint subsets with sum m by the induction hypothesis. Consequently, $\{1, 3, \ldots, 2n-1\}$ contains the desired $k+1$ disjoint subsets with sum m and $m/2$, respectively.

References

[1] Y. Alavi, A. J. Boals, G. Chartrand, P. Erdös and O. R. Oellerman, The ascending subgraph decomposition problem, *Congressus Numerantium.* **58** (1987) 7–14.

[2] K. Ando, S. Gervacio and M. Kano, Disjoint integer subsets having a constant sum, *Disctete Math.* **82** (1990) 7–11.

[3] H.L. Fu, Some results on ascending subgraph decomposition, *Bull. Inst. Math. Academia Sinica* **16** (1988) 315–319.

[4] H.L. Fu and W. H. Hu, Disjoint odd integer subsets having a constant odd sum, *Discrete Math.*, to appear.