Binding Numbers and f-Factors of Graphs

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Abstract

Let G be a connected graph of order n, a and b be integers such that $1 \leq a \leq b$ and $2 \leq b$, and $f : V(G) \to \{a, a+1, \ldots, b\}$ be a function such that $\sum(f(x); x \in V(G)) \equiv 0(\mathrm{mod}2)$. We prove the following two results: (i) If the binding number of G is greater than $(a+b-1)(n-1)/(an-(a+b)+3)$ and $n \geq (a+b)^2/a$, then G has an f-factor; (ii) if the minimum degree of G is greater than $(bn-2)/(a+b)$, and $n \geq (a+b)^2/a$, then G has an f-factor.

1 Introduction

We consider a finite graph G with vertex set $V(G)$ and edge set $E(G)$, which has neither loops nor multiple edges. For a vertex x of G, the neighborhood $N_G(x)$ of x in G is the set of vertices of G adjacent to x, and the degree $\deg_G(x)$ of x is $|N_G(x)|$. We denote by $\delta(G)$ the minimum degree of G. For a subset X of $V(G)$, let

$$N_G(X) := \cup_{x \in X} N_G(x).$$

We say that X is independent if $N_G(X) \cap X = \emptyset$. The *binding number* $bind(G)$ of G is defined by

$$bind(G) := min\{\frac{|N_G(X)|}{|X|} | \emptyset \neq X \subset V(G), N_G(X) \neq V(G)\}$$

(cf.[12]). It is trivial by the definition that $bind(G) > c$ implies that for every subset X of $V(G)$, we have $N_G(X) = V(G)$ or $|N_G(X)| > c|X|$. It is also obvious that if $bind(G) > 1$, then G is connected. Let k be a positive integer and f ba an integer-valued function defined on $V(G)$(i.e.,$f : V(G) \to \{\cdots, 0, 1, 2, \cdots\}$). Then a spanning k-regular subgraph of G is called a k-factor of G, and a spanning subgraph F of G is called a f-factor if $\deg_F(x) = f(x)$ for all $x \in V(G)$.

In this paper, we study conditions on the binding number and on the minimum degree of a graph G which guarantee the existence of an f-factor in G. We begin with some known results.

Theorem A *(Anderson[1]). If a graph G has even order and $bind(G) \geq 4/3$, then G has a 1-factor.*

Theorem B *(Woodall[12]). If $bind(G) \geq 3/2$,then G has a Hamilton cycle, in particular, G has a 2-factor.*

Recently, Katerinis and Woodall[8] and Katerinis[6] found the following sufficient conditons for a graph to have a k-factor. These conditions were also obtained by Egawa and Enomoto[3] independently.

Theorem C *Let $k \geq 2$ be an integer and G ba a graph of order n. Assume $n \geq 4k - 6$ and kn is even. Then the following two statements holds:*

(i) If $bind(G) > (2k-1)(n-1)/(kn-2k+3)$, then G has a k-factor[8].

(ii) If $\delta(G) \geq n/2$, then G has a k-factor[6].

It is shown that the conditions in (i) and (ii) are best possible. Let us note that if $k \geq 3$ and $n \geq 4k - 5$, then

$$2 - \frac{1}{k} \leq \frac{(2k-1)(n-1)}{kn-2k+3} < 2.$$

We now give our theorem, which is an extension of the above Theorem C. Moreover, the theorem gives a result concerning the following question: If $bind(G) > c \geq 2$, what factor does a graph G have?

Theorem 1 *Let G be a connected graph of order n, a and b be integers such that $1 \leq a \leq b$ and $2 \leq b$, and $f : V(G) \to \{a, a+1, \ldots, b\}$. Suppose that $n \geq (a+b)^2/a$ and $\sum_{x \in V(G)} f(x) \equiv 0 (mod\ 2)$. If one of the following three conditions is satisfied, then G has an f-factor.*

$$(i) \quad bind(G) > (a+b-1)(n-1)/(an-(a+b)+3); \tag{1}$$

$$(ii) \quad \delta(G) > (bn-2)/(a+b); \tag{2}$$

$$(iii) \quad \delta(G) \geq ((b-1)n+a+b-2)/(a+b-1) \tag{3}$$

and for every non-empty independent subset X of $V(G)$,

$$|N_G(X)| \geq \frac{(b-1)n+|X|-1}{a+b-1}. \tag{4}$$

We now show that the conditions (1) and (2) are best possible. If a graph G consists of $n(n \geq 2)$ disjoint copies of a graph H, then we write $G = nH$. The *join* $G = A + B$ has $V(G) = V(A) + V(B)$ and $E(G) = E(A) \cup E(B) \cup \{xy | x \in V(A) \text{ and } y \in V(B)\}$. Let $c = [b/a]$, m be a positive integer, and $G = K_{2mb-2m-2c} + (ma-1)K_2$, where K_l denotes the complete graph of order l. Define a function $f : V(G) \to \{a, a+1, \ldots, b\}$ by

$$f(x) = \begin{cases} a & \text{if } x \in V(K_{2mb-2m-2c}) \\ b & \text{otherwise.} \end{cases}$$

Then G has no f-factor since for $S = V(K_{2mb-2m-2c})$ and $T = V(G) \backslash S$, we have

$$\gamma_G(S,T) = 2b - 2ac - 2 < 0 \quad \text{(see Lemma 1).}$$

Moreover, we have

$$bind(G) = \frac{(a+b-1)(n-1)}{na - (a+b) + 3 + 2(ac-b)}.$$

Note that for $X = V(G) \backslash (V(K_{2mb-2m-2c}) \cup \{u\})$, where $V(K_2) = \{u, v\}$, we obtain

$$\frac{|N_G(X)|}{|X|} = \frac{n-1}{2(ma-1)-1} = \frac{(a+b-1)(n-1)}{na-(a+b)+3+2(ac-b)} = bind(G).$$

Therefore, if b is divisible by a, then condition (i) is best possible.

Next, suppose that $a + b$ is even and there exist positive integers s and t such that $bs = at + 2$ and $s + t$ is even. Let $G = (am+s)K_1 + K_{bm+t}$, where m is a positive integer, and let f be a function on $V(G)$ defined by

$$f(x) = \begin{cases} b & \text{if } x \in V((am+s)K_1), \\ a & \text{if } x \in V(K_{bm+t}). \end{cases}$$

Then G has no f-factor and

$$\delta(G) = bm + t = \frac{bn-2}{a+b}.$$

Hence condition (ii) is also best possible in this sense.

Note that (iii) of Theorem 1 is an extension of results in[9,13], which are obtained from (iii) by setting $a = b$. Similar results on 1-factor can be found in [2]. Moreover, a similar sufficient condition for a graph to have an $[a, b]$-factor, which is a spanning subgraph F such that $a \geq deg_f(x) \geq b$ for all vertices x, can be found in [5], and similar sufficient conditions for a bipartite graph to have k-factors are given in [7,4].

2 Proofs

Let G be a graph and S and T be disjoint subsets of $V(G)$. Then $G-S$ denotes the subgraph of G induced by $V(G)\backslash S$, and $e_G(S,T)$ denotes the number of edges of G joining a vertex in S to a vertex in T. Our proof of Theorem 1 is analogous to those of [3,8,9,13] and depends on the following lemma, which is called the f-factor theorem.

Lemma 1 *(Tutte[10,11]). Let G be a graph and $f: V(G) \to \{0,1,2,\cdots\}$ such that $\sum_{x\in V(G)} f(x) \equiv 0 (mod\ 2)$. Then G has an f-factor if and only if*

$$\gamma_G(S,T) := \sum_{x\in S} f(x) + \sum_{x\in T}(deg_{G-S}(x) - f(x)) - h_G(S,T) \geq 0$$

for all $S,T \subset V(G)$, $S\cap T = \emptyset$, where $h_G(S,T)$ denotes the number of components C of $G-(S\cup T)$ such that $\sum_{x\in V(C)} f(x) + e_G(V(C),T) \equiv 1 (mod\ 2)$.

Moreover, the following useful congruence expression holds:

$$\gamma_G(S,T) \equiv \sum_{x\in V(G)} f(x) \equiv 0(\text{mod}2). \tag{5}$$

Lemma 2 *[12]. Let G be a graph of order n. If $bind(G) > c$, then $\delta(G) > ((c-1)n+1)/c$, and $|N_G(X)| > ((c-1)n+|X|)/c$ for all non-empty subsets X of $V(G)$ with $N_G(X) \neq V(G)$.*

Proof. Let $Y := V(G)\backslash N_G(X)$. Since $N_G(Y) \subseteq V(G)\backslash X$, we have $n - |X| \geq |N_G(Y)| > c|Y| = c(n-|N_G(X)|)$. Hence $|N_G(X) > ((c-1)n+|X|)/c$, and so $\delta(G) > ((c-1)n+1)/c$.

Suppose that (i) in Theorem 1 holds. Then, by Lemma 2, we have

$$\delta(G) > \frac{(b-1)n+a+b-3}{a+b-1} \geq \frac{(b-1)n}{a+b-1}, \tag{6}$$

and

$$\begin{aligned}|N_G(X)| &> \frac{(b-1)n+a|X|+(b-3)(n-|X|)/(n-1)}{a+b-1} \\ &\geq \frac{(b-1)n+|X|-2}{a+b-1}\end{aligned}$$

for every independent subset X of $V(G)$. Hence G satisfies (3) and (4). Therefore (i) of Theorem 1 is an immediate consequence of (iii) of the theorem, and so we shall prove (ii) and (iii) of the theorem.

Proof of (iii) of Theorem 1. Suppose that G satisfies the conditions (3) and (4), but has no f-factor. By Lemma 1 and (5), there exists disjoint subsets S and T of $V(G)$ such that

$$\sum_{x\in S} f(x) + \sum_{x\in T}(deg_{G-S}(x) - f(x)) - w \leq -2,$$

where w denotes the number of components of $G - (S \cup T)$. Note that $S \cup T \neq \emptyset$ since $\gamma(\emptyset, \emptyset) = -h(\emptyset, \emptyset) = 0$, which follows from the assumption that G is connected and $\sum_{x\in V(G)} f(x) \equiv 0(\text{mod}2)$. In particular, we have

$$a|S| + \sum_{x\in T}(\deg_{G-S}(x) - b) - w \leq -2. \tag{7}$$

We choose S and T so that $|S|+|T|$ is as large as possible subject to $\gamma(S,T) < 0$. Let $s := |S|$ and $t := |T|$. It is clear that

$$w \leq n - s - t. \tag{8}$$

If $w > 0$ then let m denote the minimum order of components of $G-(S\cup T)$. Then

$$m \leq \frac{n-s-t}{w}. \tag{9}$$

and

$$\delta(G) \leq m - 1 + s + t. \tag{10}$$

Moreover, it follows from the choice of S and T that

$$\text{if } a = b \text{ then } m \geq 3. \tag{11}$$

(cf.[8]). If $T \neq \emptyset$, let

$$h := min\{\deg_{G-s}(x) | x \in T\}.$$

Then obviously

$$\delta(G) \leq h + s. \tag{12}$$

We consider five cases and derive a contradiction in each case.

Case 1. $T = \emptyset$. By (7) and (8), we have

$$as + 2 \leq w \leq n - s. \tag{13}$$

Hence we have by (6),(10),(9) and (12) that

$$\begin{aligned}\frac{(b-1)n}{a+b-1} &< \delta(G) \le m-1+s \le \frac{n-s}{w} - 1 + s \\ &\le \frac{n-s}{as+2} - 1 + s \\ &= \frac{n-2}{a+1} - \frac{(n-2-as-s)(as-a+1)}{(a+1)(as+2)}.\end{aligned}$$

Since $n-2-as-s \ge 0$ by (13), it follows that

$$\frac{(b-1)n}{a+b-1} < \frac{n-2}{a+1},$$

which implies $a(b-2)n < -2(a+b-1)$. This is clearly impossible since $b \ge 2$.

Case 2. $T \ne \emptyset$ and $h = 0$. Let $Z := \{x \in T \,|\, \deg_{G-S}(x) = 0\} \ne \emptyset$ and $z = |Z|$. Since Z is independent, we have by (4)

$$\frac{(b-1)n+z-1}{a+b-1} \le |N_G(Z)| \le s. \tag{14}$$

On the other hand, we have by (7), (8) and the fact that $b-1 \ge 1$

$$as - bz + (1-b)(t-z) - (b-1)(n-s-t) \le -2.$$

Hence

$$s \le \frac{(b-1)n+z-2}{a+b-1}.$$

This contradicts (14).

Case 3. $T \ne \emptyset$ and $1 \le h \le b-1$. By (7), (8), and the fact that $b-h \ge 1$, we have

$$as + (h-b)t - (b-h)(n-s-t) \le -2.$$

Thus

$$s \le \frac{(b-h)n-2}{a+b-h}. \tag{15}$$

On the other hand, we obtain by (3) and (12) that

$$\frac{(b-1)n-1}{a+b-1} + 1 \le \delta(G) \le s+h.$$

This inequality together with (15) gives us

$$\frac{(b-1)n-1}{a+b-1}+1-h \le \frac{(b-h)n-2}{a+b-h}.$$

Hence

$$(h-1)an \le (h-1)(a+b-1)(a+b-h)-(a+b+h-2).$$

This implies $h \ge 2$ and

$$an \le (a+b-1)(a+b-h)-\frac{(a+b+h-2)}{h-1}.$$

This contradicts our assumption that $n \ge (a+b)^2/a$.

Case 4. $T \ne \emptyset$ and $h = b$. We have $w \ge as+2$ by (7), and so we obtain by (9) that

$$m \le \frac{n-s-t}{w} \le \frac{n-s-1}{as+2}. \tag{16}$$

If $b \ge 3$ then we get the following inequality from $an \ge (a+n)^2 > (a+b+1)(a+b-1)$:

$$an(b-2) > (a+b-1)(ab+b^2-2a-b-2). \tag{17}$$

By (3) and (12), we have

$$\frac{(b-1)n-1}{a+b-1}+1 \le \delta(G) \le h+s = b+s$$

and so

$$\begin{aligned} s &\ge \frac{(b-1)n-1}{a+b-1}-(b-1) && (18)\\ &= \frac{n-3}{a+1}+\frac{an(b-2)+(a+b-1)(3-(a+1)(b-1))-(a+1)}{(a+b-1)(a+1)} \\ &> \frac{n-3}{a+1}+\frac{(a+b-1)(b^2-a-2b)-+2b+a-3}{(a+b-1)(a+1)}. && \text{(by(17))} \end{aligned}$$

Hence, If $b \ge 3$ then $s > (n-3)/(a+1)$, and so $m < 1$ by (16), a contradiction. If $a = b = 2$ then $s \ge (n-4)/3$ by (18), and so $m < 3$ by (16). This contradicts (11). Therefore we may assume that $a = 1$ and $b = 2$. By (16) and (18), we have $m = 1$. Thus it follows from (10) and (12) that

$$\delta(G) \le s+t \quad \text{and} \quad \delta(G) \le b+s = s+2.$$

Hence, by (7) and (8), we obtain

$$\delta(G) \le s + 2 = as + 2 \le w \le n - s - t \le n - \delta(G).$$

Hence $\delta(G) \le n/2$. This contradicts (3).

Case 5. $T \neq \emptyset$ and $h > b$. By (7), we have $as + (h-b)t - w \le -2$, and so

$$w \ge as + t + 2 \ge s + t + 2. \tag{19}$$

Suppose that $m \ge 3$. Then, by (10) and (9), we have

$$\begin{aligned} \delta(G) &\le m - 1 + s + t \le m + w - 3 \\ &\le m + w - 3 + \frac{1}{3}(m-3)(w-3) = \frac{mw}{3} \le \frac{n}{3}. \end{aligned}$$

This contradicts (4). Thus we may assume that $m \le 2$. It follows from (8) and (19) that $s + t + 1 \le n/2$. Then by (3) and (10), we have

$$\frac{(b-1)n}{a+b-1} < \delta(G) \le s + t + 1 \le \frac{n}{2}.$$

Thus $n(2b - a - 1) < 0$. This is impossible. Consequently, (iii) is proved.

Proof of (ii) of Theorem 1. This is almost identical to the proof of (iii). Since $n \ge (a+b)^2/a$, we have

$$\frac{bn-2}{a+b} \ge \frac{(b-1)n + a + b - 2}{a+b-1},$$

and so (4) still holds by (3). Thus Cases 1, 3, 4 and 5 carry over without modification from (iii) to (ii) because we don't use (4) in these cases. The only case that needs to change is the following:

Case 2. $T \neq \emptyset$ and $h = 0$. By (7) and (8), we have

$$-2 \ge as - bt - (n - s - t) \ge as - bt - b(n - s - t),$$

and so $s \le (bn-2)/(b+a)$. Then (3) and (11) give

$$\frac{bn-2}{a+b} < \delta(G) \le h + s = s \le \frac{bn-2}{a+b},$$

a contradiction. Consequently the proof is complete.

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