Disjoint subset of integers having a constant sum

Kiyoshi Ando,[1] Severino Gervacio[2] and Mikio Kano[3]
[1]Nippon Ika University, Nakahara, Kawasaki, 211, Japan
[2]MSU-Iligan Institute of Technology, Tibanga, Iligan City, Philippines
[3]Akashi Technological College, Uozumi, Akashi, 674, Japan

Abstract

We prove that for position integers n, m and k, the set $\{1, 2, \ldots, n\}$ of integers contains k disjoint subsets having a constant sum m if and only if $2k - 1 \leq m \leq n(n+1)/(2k)$.

1 Introduction

suppose that a set $\{1, 2, \ldots, n\}$ of integers contains k disjoint subsets $A_1, A_2, \ldots, A_k$ such that the sum of all elements of each A_i is equal to m. Then

$$km = \sum_{i=1}^{k} \sum_{x \in A_i} x \leq 1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Hence $m \leq n(n+1)/(2k)$. Let $x_1 < x_2 < \cdots < x_t$ be all the elements in $A_1 \cup A_2 \cup \cdots \cup A_k$. Since at least $k-1$ subsets must have at least two elements, we see that $2(k-1) + 1 \leq t$. It is clear that $t \leq x_t \leq m$. Hence $2k - 1 \leq m$. Therefore $2k - 1 \leq m \leq n(n+1)/(2k)$. Our main theorem shows that this obviously necessary condition turns out to be sufficient.

Theorem 1. *Let n, m and k be positive integers. Then the set $\{1, 2, \ldots, n\}$ of integers contains k disjoint subsets $A_1, A_2, \ldots, A_k$ such that the sum of all elements of each A_i is equal to m if and only if*

$$2k - 1 \leq m \leq \frac{n(n+1)}{2k}. \qquad (1)$$

For example, let $n = 11,\ m = 16$ and $k = 4$. Then $\{1, 2, \dots, 11\}$ contains four disjoint subsets $A_1 = \{1, 4, 11\}$, $A_2 = \{6, 10\}$, $A_3 = \{7, 9\}$, $A_4 = \{3, 5, 8\}$ with sum 16. Note that if $m = n(n + 1)/(2k)$, then the disjoint subsets $A_1, A_2, \dots, A_k$ give a partition of $\{1, 2, \dots, n\}$.

Next assume that a set $\{1, 2, \dots, n\}$ of integers has a partition $A_1 \cup A_2 \cup \cdots \cup A_k$ such that each A_i contains n/k elements and the sum of all elements of each A_i is equal to $n(n+1)/(2k)$. Then we easily observe that both n and $n(n + 1)/2$ are divisible by k, and $n/k \geq 2$. The next result shows that this necessary condition is sufficient for the existence of such a partition.

Theorem 2. *Let n and k be positive integers such that n is divisible by k. Then the set $\{1, 2, \dots, n\}$ of integers has a partition $A_1 \cup A_2 \cup \cdots \cup A_k$ such that each A_i contains n/k elements and the sum of all elements of each A_i is equal to $n(n+1)/(2k)$ if and only if $n(n+1)/2$ is divisible by k, and $n/k \geq 2$.*

Note that this result can be extended to an arbitrary arithmetic progression by only replacing each integer i by the ith term $a - d + id$. That is, an arithmetic progression $\{a, a + d, a + 2d, \dots, a + (n - 1)d\}$ has a partition $A_1 \cup A_2 \cup \cdots \cup A_k$ such that each A_i contains n/k elements and has the constant sum $n(2a + (n - 1)d)/(2k)$ if and only if both n and $n(n + 1)/2$ is divisible by k, and $n/k \geq 2$.

We conclude this section by giving a conjecture.

Conjecture. *Let n, m and k be positive integers. Then the set $\{1, 3, \dots, 2n-1\}$ of odd integers contains k disjoint subsets having a constant sum m if and only if one of the following two conditions holds:*

(i) m is even, $4k \leq m \leq n^2/k$, $n^2 - mk \neq 2$, and $m \neq 4n - 2$ or $n \neq 4k$.

(ii) m is odd, and $9(k - 1) \leq m \leq 2n - 1$, or $9k \leq m \leq n^2/k$ and $n^2 - mk \neq 2$.

It is easy to see that these conditions are necessary. Note that Conjecture (1) has been proved by H. Enomoto and M. Kano. This result will be treated in another paper.

2 Proofs of theorems

For convenience, we use the following notation for an integer valued function f defined on a set X.

$$\sum (f(x); x \in X) = \sum_{x \in X} f(x).$$

We first prove the sufficiency of Theorem 1. Note that necessity in the theorem was discussed in the preceding section.

Proof of Theorem 1. We prove the sufficiency by induction on n. We may assume $k \geq 2$ since the theorem is true for $k = 1$. Verification is easy for $n = 2, 3, 4, 5$. Let $n = 6$. Observe that the condition $2k - 1 \leq n(n+1)/(2k)$ implies $2k - 1 \leq n$. If $m \leq n$, then we can obtain the desired k disjoint subsets $A_1 = \{m\}$, $A_2 = \{1, m-1\}$, ..., $A_k = \{k-1, m-k+1\}$, where $k-1 < m-k+1$ by (1). If $m = n+1$, then $2k \leq n$ by (1), and so $A_1 = \{1, n\}$, $A_2 = \{2, n-1\}$, ..., $A_k = \{k, n-k+1\}$ are k disjoint subsets with sum m. If $n \leq 2k$, then $m \leq n+1$ by (1), and thus we can get k disjoint subsets of $\{1, 2, \ldots, n\}$ with sum m as above. Therefore, we may assume

$$n + 2 \leq m \quad \textit{and} \quad 2k + 1 \leq n. \tag{2}$$

We consider four cases.

Case 1. $2n \leq m$.

Let $L_1 = \{n-2k+1, n\}$, $L_2 = \{n-2k+2, n-1\}$,...,$L_k = \{n-2k+k, n-k+1\}$. Then $\sum(x; x \in L_i) = 2n - 2k + 1$ for all i, $1 \leq i \leq k$. Let $m' = m - 2n + 2k - 1$ and $n' = n - 2k$. Then we show that m' and n' are positive integers which satisfy the following condition.

$$2k - 1 \leq m' \leq n'(n'+1)/(2k). \tag{3}$$

Since $2k + 1 \leq n$, $2 \leq k$ and $2n \leq m$, we have $n' > 0$, $m' > 0$ and $2k - 1 \leq m'$. It follows that

$$\begin{aligned} m'k &= mk - k(2n - 2k + 1) \leq n(n+1)/2 - \sum_{i=1}^{k}(\sum(x; x \in L_i)) \\ &= 1 + 2 + \cdots + n - ((n'+1) + (n'+2) + \cdots + n) \\ &= 1 + 2 + \cdots + n' = n'(n'+1)/2. \end{aligned}$$

Hence $m' \leq n'(n'+1)/(2k)$. Consequently, (3) holds. By the induction hypothesis, the set $\{1, 2, \ldots, n'\}$ contains k disjoint subsets $M_1, M_2, \ldots, M_k$ with sum m'. Then $L_1 \cup M_1$, $L_2 \cup M_2$,...,$L_k \cup M_k$ are k disjoint subsets of $\{1, 2, \ldots, n\}$ as required.

Case 2. $m < 2n$ and $4k \leq n$.

Suppose $m \leq 2n - 2$. Then $2k - 1 \leq m \leq (n-1)n/(2k)$ since $m \leq 2(n-1) = 4k(n-1)/(2k) \leq n(n-1)/(2k)$. By the induction hypothesis, we can obtain k disjoint subsets with sum m from $\{1, 2, \ldots, n-1\}$. Thus we may assume $m = 2n - 1$. Let $n' = n - 2$ and $k' = k - 1$. Then

$2k^{'} - 1 \leq m \leq n^{'}(n^{'}+1)/(2k^{'})$ since the former inequality follows from (1), and the latter follows from

$$\begin{aligned} mk^{'} &= mk - m \leq n(n+1)/2 - (n-1+n) \\ &= 1 + 2 + \cdots + n^{'} = n^{'}(n^{'}+1)/2. \end{aligned}$$

Therefore, the set $\{1, 2, \ldots, n-2\}$ contains $k-1$ disjoint subsets $A_1, A_2, \ldots, A_{k-1}$ with sum m. By adding a subset $A_k = \{n-1, n\}$, we have k disjoint subsets of $\{1, 2, \ldots, n\}$ with sum m.

Case 3. $m < 2n$, $n < 4k$ and m is odd.
Put $t = (2n - m + 1)/2$. Then t is a positive integer. Let $A_1 = \{m-n, n\}$, $A_2 = \{m-n+1, n-1\}$, ..., $A_t = \{m-n+t-1, n-t+1\}$. Then $A_1 \cup A_2 \cup \cdots \cup A_t = \{m-n, m-n+1, \ldots, n\}$ since $m-n+t = n-t+1$. If $k \leq t$ we are done, so we assume $t < k$. Let $k^{'} = k-t$ and $n^{'} = m-n-1$. Then $n^{'} > 0$ by (2), and $2k^{'} - 1 \leq m$ by (1). The inequality $m \leq n^{'}(n^{'}+1)/(2k^{'})$ follows from

$$\begin{aligned} mk^{'} &= mk - mt \leq n(n+1)/2 - \sum_{i=1}^{t} \sum (x; x \in A_i) \\ &= 1 + 2 + \cdots + n^{'} = n^{'}(n^{'}+1)/2. \end{aligned}$$

Therefore, $2k^{'} - 1 \leq m \leq n^{'}(n^{'}+1)/(2k^{'})$. By the induction hypothesis, $\{1, 2, \ldots, n^{'}\}$ contains $k-t$ disjoint subsets $B_1, B_2, \ldots, B_{k-t}$ with sum m. Consequently, we have desired k is disjoint subsets $A_1, \ldots, A_t, B_1, \ldots, B_{k-t}$.

Case 4. $m < 2n$, $n < 4k$ and m is even.
Put $t = (2n - m)/2$. Then t is a positive integer. Let $A_1 = \{m-n, n\}$, $A_2 = \{m-n+1, n-1\}$, ..., $A_t = \{m-n+t-1, n-t+1\}$. Then $A_1 \cup A_2 \cup \cdots \cup A_t = \{m-n, m-n+1, \ldots, n\} \backslash \{m/2\}$ since $m-n+t = n-t = m/2$. If $k \leq t$ we are done, so we assume $t < k$. Let $k^{'} = 2(k-t) - 1$ $n^{'} = m-n-1$ and $m^{'} = m/2$. Then $k^{'}$, $n^{'}$ and $m^{'}$ are positive integers, and we shall show that the following condition holds.

$$2k^{'} - 1 \leq m^{'} \leq n^{'}(n^{'}+1)/(2k^{'}). \tag{4}$$

Assume (4) holds. Then $\{1, 2, \ldots, n^{'}\}$ contains $k^{'}$ disjoint subsets $L_1, L_2, \ldots, L_{k'}$ with sum $m^{'}$. Therefore, we obtain k disjoint subsets $A_1, \ldots, A_t, L_1 \cup L_2, L_3 \cup L_4, \ldots, L_{k'-2} \cup L_{k'-1}, L_{k'} \cup \{m/2\}$ with sum m from $\{1, 2, \ldots, n\}$.

We now prove (4). The inequality $2k^{'} - 1 \leq m^{'}$ holds if and only if $m \leq (8n - 8k + 6)/3$. Since $m \leq n(n+1)/(2k)$, the inequality $2k^{'} - 1 \leq m^{'}$ holds if $n(n+1)/(2k) \leq (8n - 8k + 6)/3$. The last inequality is equivalent to

$(4k-3)/3 \le n \le 4k$. By the assumption of this case, this inequality holds. The inequality $m' \le n'(n'+1)/(2k')$ follows from

$$\begin{aligned} m'k' &= mk - (mt + m/2) \le n(n+1)/2 - \sum_{i=1}^{t} \sum (x; x \in A_i) - m/2 \\ &= 1 + 2 + \cdots + n' = n'(n'+1)/2. \end{aligned}$$

This completes the proof of the theorem. □

We next prove Theorem 2.

Proof of Theorem 2. We shall prove only the sufficiency because the necessity is shown in Section 1, Let r be any positive integer. Then a set $\{r+1, r+2, \ldots, r+2k\}$ of integers can be partitioned into the desired k subsets $\{r+1, r+2k\}$,$\{r+2, r+2k-1\}$,..., $\{r+k, r+k+1\}$ with sum $2r+2k+1$. Hence, if n is divisible by $2k$, then the set $\{1, 2, \ldots, n\}$ can be partitioned into desired k disjoint subsets since $\{1, 2, \ldots, n\}$ can be partitioned into disjoint subsets $\{2ki+1, 2ki+2, \ldots, 2ki+2k\}$, $0 \le i \le (n/2k)-1$. Therefore we may assume that n is not divisible by $2k$. Then k must be odd since $n(n+1)/2$ is divisible by k. Put $k = 2t+1$. We now show that a set $\{1, 2, \ldots, 3k\} = \{1, 2, \ldots, 6t+3\}$ can be partitioned into k disjoint subsets having constant sum, and with three elements each. For every i, $1 \le i \le t$, define

$$A_{2i-1} = \{i, 3t+1+i, 6t+5-2i\},$$
$$A_{2i} = \{t+i+1, 2t+1+i, 6t+4-2i\},$$

and set

$$A_{2t+1} = \{t+1, 4t+2, 4t+3\}.$$

Then $A_1, A_2, \ldots, A_{2t+1}$ give a desired partition of $\{1, 2, \ldots, 3k\}$, Since $n/k \ge 3$, we can partition $\{1, 2, \ldots, n\}$ into $\{1, 2, \ldots, 3k\}$ and $\{3k+2ki+1, \ldots, 3k+2ki+2k\}$, $0 \le i \le (n-3k)/(2k)-1$. Each subset can be partitioned into k disjoint subsets with constant sum and of constant cardinality, and so we can conclude that the set $\{1, 2, \ldots, n\}$ can be partitioned into desired k disjoint subsets. □