Sufficient conditions for a graph to have factors

Mikio Kano
Department of mathematics, Akashi Technologyical College
Uozumi, Akashi 674, Japan

Abstract

We present sufficient conditions for a graph to have an f-factor or a (g, f)-factor that contains given edges or does not contain any given edges, where g, and f be integer-valued functions defined on the vertices of the graph.

1 Introduction

We consider finite graphs which may have muitiple edges but have no loops. All notation and definitions not given here can be found in [3] or [5].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex v of a subgraph H of G, we denote by $\deg_H(v)$ the degree of v in H. Let g and f be integer-valued functions defined on $V(G)$. Then an f-factor of G is a spanning subgraph F of G such that $\deg_F(x) = f(x)$ for every vertex x of G. A spanning subgraph H of G satisfying $g(x) \leq d_F(x) \leq f(x)$ for all $x \in V(G)$ is called a (g, f)-factor of G. Then an f-factor and a (g, f)-factor with $g \equiv f$ are the same.

Let r be a positive integer. Then an r-regular graph G satisfies $d_G(x) = r$ for every $x \in V(G)$, and a spanning subgraph F is called an r-factor(an r-regular factor) if $d_F(x) = r$ for all $x \in V(F)$.

A criterion for the existence of an f-factor was found by Tutte[11], and Lovàsz[10] gave a necessary and sufficient condition for a graph to have a (g, f)-factor. Simple sufficient conditions for a graph to have an f-factor and a (g, f)-factor are given in [8] and [9], and these conditions include edge-connectivity of a graph. In this paper, we give sufficient conditions for a graph to have an f-factor and a (g, f)-factor that contain given edges or do not contain given edges. For factors of graphs, the reader should refer to a survey [1].

2 Theorems

Let us first give some known results related to our theorems. In order to do so, we need some notation. By $|X|$, we denote the cardinality of a set X. For two disjoint subsets S and T of $V(G)$, we denote by $e_G(S,T)$ the number of edges of G joining S to T. For a non-empty proper subset X of $V(G)$, we write $\partial(X) = \partial_G(X) = e_G(X, V(G)/X)$. When we deal with $\partial(X)$, we always assume $\emptyset \neq X \neq V(G)$. The order of G is $|V(G)|$. We use the following notation:

$$\sum(f(x); x \in X) = \sum_{x \in X} f(x) \quad \text{and} \quad \sum(f(x_i); 1 \leq i \leq n) = \sum_{i=1}^{n} f(x_i).$$

In the following propositions, let k, n and r denote positive integers.

Proposition 1 (Bäbler[2]) . *Let G be a connected r-regular graph of even order and $n \geq 2$. Suppose $\partial(X) \geq n$ for all $X \subset V(G)$ with $|X|$ odd. If r and k are odd and $r/n \leq k$, then G has a k-factor.*

Proposition 2 (Gallai[6]) . *Let G be a connected r-regular graph. Suppose $\partial(X) \geq n$ for all $X \subset V(G)$ with $|X|$ odd. Then*

(1) If k is odd, $|V(G) \equiv 0(mod\ 2)$, and $r/n \leq k \leq r(n-1)/n$, then G has a k-factor; and

(2) If k is even and $2 \leq k \leq r(n-1)/n$, then G has a k-factor.

Proposition 3 (Bermond and Las Vergnas[4]) . *Let G be a connected graph of even order, and k be an odd number. Suppose $\partial(X) \geq r/k$ for all $X \subset V(G)$ with $|X|$ odd or $\partial(X)$ odd. If $1 \leq k \leq r/2$ and $\sum(|d_G(x)-r|; x \in V(G)) < 2r/k$, then G has a k-factor.*

Note that the condition $\sum(|d_G(x) - r|; x \in V(G)) < r/k$ in [4] can be replaced by $\sum(|d_G(x) - r|; x \in V(G)) < 2r/k$ as above. It is obvious that the condition $\partial(X) \geq n$ holds if G is n-edge-connected. For a vertex subset X of a graph G, we denote by $G[X]$ the subgraph of G induced by X, and by $G - X$ the subgraph of G obtained from G by deleting the vertices in X together with their incident edges. We now give our theorems; one is a result on f-factors and the other is a result on (g, f)-factors.

Theorem 1 *Let G be a connected graph, θ be a real number such that $0 \leq \theta \leq 1$, A and B be disjoint subsets of $E(G)$, and f be an integer-valued function defined on $V(G)$. If the following four conditions hold, then G has an f-factor F such that $E(F) \supset A$ and $E(F) \cap B = \emptyset$.*

(1) $\sum(f(x); x \in V(G)) \equiv 0(mod\ 2)$.
(2) $\varepsilon = \sum(|f(x) - \theta d_G(x)|; x \in V(G)) + 2(1-\theta)|A| + 2\theta|B| < 2$.
(3) $\theta\partial(X) \geq 1$ *for all* $X \subset V(G)$ *such that* $G[X]$ *is connected and*

$$\sum(f(x); x \in X) \equiv 1(mod\ 2). \tag{2.1}$$

(4) $(1-\theta)\partial(X) \geq 1$ *for all* $X \subset V(G)$ *such that* $G[X]$ *is connected and*

$$\sum(f(x); x \in X) + \partial(X) \equiv 1(mod\ 2). \tag{2.2}$$

Note that $\emptyset \neq X \neq V(G)$ in (3) and (4) because we always assume so when we deal with $\partial(X)$. Furthermore, at least one of A and B must be an empty set by (2). We give some remarks on Theorem 1, which are useful for applications of Theorem 1.

Lemma 1 *Let* G, f *and* X *be the same as in Theorem 1. Then the following statements hold.*
(1) $\sum(d_G(x); x \in X) \equiv \partial(X)(mod\ 2)$. (2.3)
(2) If $\{f(x)|x \in V(G)\}$ *consists of even numbers, then (3) of Theorem 1 holds.*
(3) If both $\{f(x)|x \in V(G)\}$ *and* $\{d_G(x)|x \in V(G)\}$ *consists of even numbers, then (3) and (4) of Theorem 1 hold.*
(4) If both $\{f(x)|x \in V(G)\}$ *and* $\{d_G(x)|x \in V(G)\}$ *consists of odd numbers, then (4) of Theorem 1 holds.*
(5) If G *is a regular graph and* $f(x) = K$ *for all* $x \in V(G)$*, then (2.2) implies* $|X| \equiv 1(mod\ 2)$.

Proof. (2.3) follows at once from $\sum(d_G(x); x \in X) = 2|E(G[X])| + \partial(X)$. We next prove (4). Since $\{f(x)|x \in V(G)\}$ consists of odd numbers, we have by (2.2) that $\sum\{f(x); x \in X) + \partial(X) \equiv |X| + \partial(X) \equiv 1(\text{mod } 2)$. On the other hand, we obtain $\sum(d_G(x); x \in X) \equiv |X| \equiv \partial(X)(\text{mod } 2)$ by the assumption on $\{d_G(x)|x \in V(G)\}$ and (2.3), a contradiction. Therefore there is no X satisfying (2.2), and thus (4) of Theorem 1 holds. Statements (2), (3) and (5) can be proved similarly. □

Before giving the other theorem, we show that the propositions mentioned previously can be obtained by making use of Theorem 1.

Proof of Proposition 1. Set $f(x) = k$ for all $x \in V(G)$, $A = B = \emptyset$, and $\theta = k/r$. We show that the conditions in Theorem 1 hold. It is obvious that (1) and $(2; \varepsilon = 0)$ hold. Since k is odd, we have by (2.1) that $|X| \equiv \sum(f(x); x \in X) \equiv 1(\text{mod } 2)$, and thus $\partial(X) \geq n$. Hence

$\theta\partial(X) \geq (k/r)n \geq 1$, and so (3) holds. It follows from (4) of Lemma 1 that (4) is true. Consequently G has a k-factor. $\Box$

Proof of Proposition 2. Set $f(x) = k$ for all $x \in V(G)$, $A = B = \emptyset$, and $\theta = k/r$. We prove that the conditions in Theorem 1 are satisfied. It is clear that (1) and $(2; \varepsilon = 0)$ hold. Suppose first k is odd. Then (2.1) implies $|X| \equiv 1(\text{mod } 2)$, and so $\theta\partial(X) \geq (k/r)n \geq 1$ as $\partial(X) \geq n$ and $r/n \leq k$. Hence (3) follows. By (5) of Lemma 1, (2.2) implies $|X| \equiv 1(\text{mod } 2)$. Thus $(1-\theta)\partial(X) \geq (1-k/r)n \geq 1$ as $k \leq r(n-1)/n$. Therefore (4) holds and we conclude that G has a k-factor. We next assume k is even. By (2) of Lemma 1, (3) holds. By (5) of Lemma 1, we have $|X| \equiv 1(\text{mod } 2)$ and so $(1-\theta)\partial(X) \geq (1-k/r)n \geq 1$ as $k \leq r(n-1)/n$. Hence (4) follows, and thus G has a k-factor. $\Box$

Proof of Proposition 3. Set $f(x) = k$ for all $x \in V(G)$, $A = B = \emptyset$, and $\theta = k/r$. We show that the conditions in Theorem 1 hold. Since G is of even order, (1) holds. Since k is odd, we have $|X| \equiv 1(\text{mod } 2)$ by (2.1), and so $\theta\partial(X) \geq (k/r)\cdot(r/k) = 1$. It is immediate that (2.2) implies that exactly one of $|X|$ and $\partial(X)$ is odd. Hence $(1-\theta)\partial(X) \geq (1-k/r)(r/k) \geq 1$ as $k \leq r/2$. Moreover, (2) follows from $\varepsilon = (k/r)\sum(|\deg_G(x)-r|; x \in V(G)) < (k/r)\cdot(2r/k) = 2$. Consequently G has a k-factor. $\Box$

The other theorem is the following.

Theorem 2 *Let G be a connected graph, θ be a real number such that $0 \leq \theta \leq 1$, A and B be disjoint subsets of $E(G)$, and g and f be integer-valued functions defined on $V(G)$ satisfying $g(x) \geq f(x)$ for all $x \in V(G)$. If the following four conditions hold, then G has a (g,f)-factor F such that $A \subset E(F)$ and $E(F) \cap B = \emptyset$.*

(1) There exists at least one vertex v satisfying $g(v) < f(v)$.

(2) $\varepsilon = \sum(\max\{0, g(x) - \theta d_G(x)\} + \max\{0, \theta d_G(x) - f(x)\}; x \in V(G)) + 2(1-\theta)|A| + 2\theta|B| < 1$.

(3) $\theta\partial(X) \geq 1$ for all $X \subset V(G)$ such that $G[X]$ is connected, $g(x) = f(x)$ for all $x \in X$ and $\sum(f(x); x \in X) \equiv 1(\text{mod } 2)$.

(4) $(1-\theta)\partial(X) \geq 1$ for all $X \subset V(G)$ such that $G[X]$ is connected, $g(x) = f(x)$ for all $x \in X$ and $\sum(f(x); x \in X) + \partial(X) \equiv 1(\text{mod } 2)$.

We next give a corollary of Theorem 2.

Proposition 4 (Little, Grant and Hoton) . *Let G be a $2r$-edge-connected $2r$-regular graph of odd order. Then for all any vertex v of G, $G - v$ has a 1-factor that contains no $r-1$ given edges.*

Proof. Define two functions g and f on $V(G)$ by $g(x) = f(x) = 1$ for all $x \in V(G)/\{v\}$, $g(v) = 0$ and $f(v) = 1$. Set $\theta = 1/(2r)$, $A = \emptyset$ and $B \subset E(G)$ with $|B| = r-1$. We prove that the conditions in Theorem 2 hold. (1) follows ar once from $g(v) < f(v)$. Since $\varepsilon = 2\theta|B| = (r-1)/r < 1$, (2) is satisfied. Ir follows from $\partial(X) \geq 2r$ that $\theta\partial(X) \geq 1$ and $(1-\theta)\partial(X) \geq 1$. Thus (3) and (4) hold. Consequently, G has a (g, f)-factor F such that $E(F) \cap B = \emptyset$, and it is easy to see that $d_F(v) = 0$. Therefore F is a desired 1-factor of $G - v$. $\Box$

3 Proofs of theorems

The following lemma plays an important role.

Lemma 2 (Lov*á*sz*′*s (g, f)-factor theorem[10]) *Let G be a connected graph and g and f be integer-valued functions defined on $V(G)$ such that $0 \leq g(x) \leq f(x) \leq d_G(x)$ for all $x \in V(G)$. Then G has a (g, f)-factor if and only if*

$$\delta(S,T) = \sum(d_G(t) - g(t); t \in T) + \sum(f(s); s \in S) - e_G(S,T) - h(S,T) \geq 0$$

for all disjoint subsets S and T of $V(G)$, where $h(S,T)$ is the number of components C of $G - (S \cup T)$ such that $g(x) = f(x)$ for all $x \in V(C)$ and $\sum(f(x); x \in V(C)) + e_G(T, V(C)) \equiv (mod\ 2)$.

Note that we can find an elegant short proof of the lemma in [13]. A necessary and sufficient condition for a graph to have an f-factor, so called Tutte$'$s f-factor theorem([11], [12]), is obtained from the above lemma by setting $g \equiv f$.

Proof of Theorem 2. Put $A = \{a_1, \ldots, a_p\}$ and $B = \{b_1, \ldots, b_q\}$, where $p = 0$ or $q = 0$ if $A = \emptyset$ or $B = \emptyset$. We first construct a new graph H from G by inserting new vertices v_i and w_j of degree 2 into edges a_i and b_j, respectively, where $1 \leq i \leq p$ and $1 \leq j \leq q$. Then $V(H) = V(G) \cup \{v_i, \ldots, v_p\} \cup \{w_1, \ldots, w_q\}$. We define two new functions g' and f' on $V(H)$ by

$$\begin{aligned} &g'(x) = g(x) \ \text{ and } \ f'(x) = f(x) \ \text{ if } \ x \in V(G), \\ &g'(x) = f'(x) = 2 \ \text{ if } \ x \in \{v_1, \ldots, v_p\}; \ \text{ and} \\ &g'(x) = f'(x) = 0 \ \text{ if } \ x \in \{w_1, \ldots, w_q\}. \end{aligned}$$

Then it is obvious that G has a (g, f)-factor F satisfying $A \subset E(F)$ and $E(F) \cap B = \emptyset$ if and only if H has a (g', f')-factor. Thus it suffices to show that H, g' and f' satisfy the conditions in Lemma 2.

Let S and T be any disjoint subsets of $V(H)$. If $S \cup T = \emptyset$, then $\delta(S,T) = \delta(\emptyset,\emptyset) = -h(\emptyset,\emptyset) = 0$ by (1) of Theorem 2. Hence we may assume $S \cup T \neq \emptyset$. Let $C_1, \ldots, C_r$ be the components of $H-(S\cup T)$ which satisfy the conditions on $h(S,T)$ in Lemma 2, where $r = h(S,T)$. Then we have

$$\begin{aligned}\delta(S,T) =&(1-\theta)\sum(d_H(t); t\in T) + \theta\sum(d_H(s); s\in S)\\ &-\sum(g^{'}(t)-\theta d_H(t); t\in T) - \sum(\theta d_H(s) - f^{'}(s); s\in S)\\ &- e_H(S,T) - r.\end{aligned}$$

Since $g^{'}(v) - \theta d_H(v_i) = 2(1-\theta)$, $g^{'}(w_j)-\theta d_H(w_j) \leq 0$, $\theta d_H(v_i) - f^{'}(v_i) \leq 0$ and $\theta d_H(w_j) - f^{'}(w_j) = 2\theta$, we obtain

$$\begin{aligned}&\sum(g^{'}(t)-\theta d_H(t); t\in T) - \sum(\theta d_H(s) - f^{'}(s); s\in S)\\ \leq&\sum(\max\{0, g^{'}(x)-\theta d_H(x)\} + \max\{0, \theta d_H(x) - f^{'}(x)\}; x\in V(H))\\ =&2(1-\theta)|A| + 2\theta|B| + \sum(\max\{0, g(x) - \theta d_H(x)\}\\ &+\max\{0, \theta d_H(x) - f(x)\}; x \in V(G))\\ =&\varepsilon.\end{aligned}$$

Hence

$$\begin{aligned}\delta(S,T) \geq&(1-\theta)\{e_H(T,S) + \sum(e_H(T,V(C_i)); 1\leq i\leq r)\}\\ &+\theta\{e_H(S,T) + \sum(e_H(S,V(C_i)); 1\leq i\leq r)\} - \varepsilon - e_H(S,T) - r\\ =&\sum((1-\theta)e_H(T,V(C_i) + \theta(S,V(C_I)) = 1; 1\leq i\leq r) - \varepsilon.\end{aligned}$$

Since $\delta(S,T)$ is an integer, $\delta(S,T) > -1$ implies $\delta(S,T) \geq 0$. Therefore, since $\varepsilon < 1$, it suffices to show that $(1-\theta)e_H(T,V(C_i)) + \theta e_H(S,V(C_i)) - 1 \geq 0$ for all i. For any $C \in \{C_i, \ldots, C_r\}$, put

$$\Delta(C) = (1-\theta)e_H(T,V(C)) + \theta e_H(S,V(C)) - 1.$$

If $e_H(T,V(C)) \geq 1$ and $e_H(S,V(C)) \geq 1$, then $\Delta \geq 0$, and thus we may assume $e_H(T,V(C)) = 0$ or $e_H(C,V(C)) = 0$. Suppose $V(C) = \{v_i\}$(or $= \{w_j\}$). Then it follows from the conditions on C that $\sum(f^{'}(x); x\in V(C)) + e_H(T,V(C)) \equiv e_H(T,V(C)) \equiv 1 (\mathrm{mod}\ 2)$. Hence $e_H(T,V(C)) = 1$ and so $e_H(S,V(C)) = 1$ a contradiction. Therefore, C contains at least one vertex of $V(G)$. We consider two cases.

Case 1. $e_H(S,V(C)) = 0$. We shall prove $\Delta(C) = (1-\theta)e_H(T,V(C)) - 1 \geq 0$. We first show that we may assume T contains neither v_i nor w_j

such that $e_H(v_i, V(C)) = 2$ or $e_H(w_j, V(C)) = 2$. Suppose T contains such a vertex v_i(or w_j). Let $T_1 = T/\{v_i\}(= T/\{w_j\})$ and $C_1 = H[V(C) \cup \{v_i\}](= H[V(C) \cup \{w_j\})$. Then $e_H(T, V(C)) = e_H(T_1, V(C_i)) + 2$, and so $(1-\theta)e_H(T_1, V(C_1)) \geq 1$ implies $(1-\theta)e_H(T, V(C)) \leq 1$. Moreover, $C = 1$ is a component of $H - (S \cup T_1)$ which satisfied the same conditions as C, that is, $g^{'}(x) = f^{'}(x)$ for all $x \in V(C_1)$, $\sum(f^{'}; x \in V(C_1)) + e_H(T_1, V(C_1)) \equiv 1(\text{mod } 2)$. Consequently, we may consider T_1 instead of T, that is , we may assume T has neither v_i nor w_j with $e_H(v_i, V(C)) = 2$ or $e_H(w_j, V(C)) = 2$.

Put $X = V(C) \cap V(G)$. We show that $g(x) = f(x)$ for all $x \in X$, $e_H(T, V(C)) = \partial_G(X)$, $G[X]$ is connected, and $\sum(f(x); x \in X) + \partial_G(X) \equiv 1(\text{mod } 2)$, which implies $\Delta(C) = (1-\theta)\partial_G(X) - 1 \geq 0$ by (4) of Theorem 2. Since $g^{'}(u) = f^{'}(u)$ for all $u \in V(C)$, it is trivial that $g(x) = f(x)$ for all $x \in X$. It follows that $\sum(f^{'}(u); u \in V(C)) = \sum(f(x); x \in X) + \sum(f^{'}(y); y \in V(C)/X) \equiv \sum(f(x); x \in X)(\text{mod } 2)$ and $e_H(T, V(C)) = \partial_H(V(C)) = \partial_G(X)$. Then we have $\sum(f(x); x \in X) + \partial_G \equiv 1(\text{mod } 2)$ by $\sum(f^{'}(u); u \in V(C)) + e_H(T, V(C)) \equiv 1(\text{mod } 2)$. Moreover, it is immediate that $G[X]$ is connected. Consequently, $\Delta(C) \geq 0$ in this case.

Case 2. $e_H(T, V(C)) = 0$. By the same argument as in case 1, we may assume that S contains neither v_i nor w_j such that $e_H(v_i, V(C)) = 2$ of $e_H(w_j, V(C)) = 2$. Let $X = V(C) \cap V(G)$. Then we can prove in the same way in the proof of case 1 that $e_H(S, V(C)) = \partial_H(V(C)) = \partial_G(X)$, $g(x) = f(x)$ for all $x \in X$, $G[X]$ is connected, and $\sum(f(x); x \in X) \equiv 1(\text{mod } 2)$. Consequently, $\Delta(C) = \theta e_H(S, V(C)) = 1 = \theta\partial_G(X) - 1 \geq 0$ by (3) of Theorem 2, and we conclude that the proof of this theorem is complete. □

Proof of Theorem 1. We first construct a new graph H and define a new function $f^{'}$ on $V(H)$ as in the proof of Theorem 2. By Tutte′s f-factor theorem. it suffices to prove that for any disjoint subsets S and T of $V(H)$,

$$\delta(S,T) = \sum(d_H(t) - f^{'}(t); t \in T) + \sum(f^{'}(s); s \in S) - e_H(S,T) - h(S,T) \geq 0.$$

If $S \cup T = \emptyset$, then $\delta(S,T) = -h(\emptyset, \emptyset) = 0$ by (1) of Theorem 1. If $S \cup T \neq \emptyset$, then we can show that $\delta(S,T) \geq -\varepsilon$ by the same argument as in the proof of Theorem 2. On the other hand, we have $\delta(S,T) \equiv 0(\text{mod } 2)$ since $\delta(S,T) \equiv \sum(f^{'}(x); x \in V(H))(\text{mod } 2)$([11, 12]) and $\sum(f^{'}(x); x \in V(H))0(\text{mod } 2)$ by (1) of Theorem 1. Therefore, if $\varepsilon < 2$, then $\delta(S,T) \geq 0$. Consequently, G has a desired f-factor. □

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