A sufficient condition for a graph to have $[a, b]$-factors

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Abstract

We give a sufficient condition by using neighborhoods for a graph to have $[a, b]$-factors.

1 Introduction

We deal with finite graphs with have neither loops nor muitiple edges. All notation and definitions not given here can be found in [4]. A list of results related to our theorems can be found in s survey article[1].

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex v of a subgraph H of G, we denote the degree of v in H by $\deg(v)$. Let a and b be integers such that $0 \leq a \leq b$. A graph G is called an $[a, b]$-graph if $a \leq \deg(x) \leq b$ for all $x \in V(G)$, and an $[a, b]$-subgraph can be defined analogously. A spanning $[a, b]$-subgraph is called an $[a, b]$-factor. An $[r, r]$-factor is usually called an r-factor. An edge jioning a vertex x to a vertex y is denoted by xy or yx. For a vertex v, we denote the neighborhood of v by $\Gamma(v) = \Gamma_G(v)$, and for a vertex subset X of G, we define the neighborhood of X an follows:

$$\Gamma(X) = \Gamma_G(X) := \cup_{x \in X} \Gamma(x).$$

Note the $\Gamma(v)$ does not contain v, but it may happen that $X \subseteq \Gamma(X)$. For a real number t, the greatest integer less than or equal to t is denoted by $\lfloor t \rfloor$. For a finite set $\mathtt{Z}$, we write $|\mathtt{Z}|$ or $\#\mathtt{Z}$ for the cardinality of $\mathtt{Z}$.

We now give some results concerning our theorems. Anderson[3] proved that if $|\Gamma(X)| \geq \frac{4}{3}|X|$ or $\Gamma(X) = V(G)$ for all $X \subset V(G)$, then the graph G has a 1-factor. Woodall[10] showed that if $|\Gamma(X)| \geq \frac{3}{2}|X|$ or $\Gamma(X) = V(G)$ for all $X \subset V(G)$, then the graph has a hamilton cycle. Moreover similar results can be found in [6] and [7]. The following theorem was obtained by Berge and Las Vergnas[5], and by Amahashi and Kano[2] independently.

Theorem 1 *Let $b \geq 2$ be an integer, Then a graph G has a $[1, b]$-factor if and only if*

$$|\Gamma(X)| \geq \frac{|X|}{b}$$

for all independent subset X of $V(G)$.

We now present our main theorem theorem, which will be proved in the next section.

Theorem 2 *Let a and b be integers such that $2 \leq a < b$, and let G be a graph of order n with $n \geq 6a + b$. Put $\lambda = (a-1)/b$. Suppose for any subset $X \subset V(G)$, we have*

$$\Gamma(X) = V(G) \quad \text{if } |X| \geq \lfloor \frac{n}{1+\lambda} \rfloor; \quad \text{or}$$
$$|\Gamma(X)| \geq (1+\lambda)|X| \quad \text{if } |X| \leq \lfloor \frac{n}{1+\lambda} \rfloor.$$

Then G has an $[a, b]$-factor.

We shall show that the condition in Theorem 2 cannot be replaced by the condition that $\Gamma(X) = V(G)$ or $|\Gamma(X)| \geq (1+\lambda)|X|$ for all $X \subset V(G)$(see Theorem 3). So the condition in Theorem 2 is best possible in this sense.

2 Proof of Theorem 2

We denote by $\delta(G)$ the minimum degree of a graph G. For any vertex subsets A and B of G, we write

$$e_G(A, B) = e(A, B) = \#\{(a, b) \in A \times B | ab \in E(G)\}.$$

The next lemma, which plays an important role, is an immediate corollary of the (g, f)-factor theorem due to Lov*á*z[8], whose short proof can be found in Tutte[9].

Lemma 1 *Let a and b be integers such that $1 \leq a < b$. Then a graph G has an $[a, b]$-factor if and only if*

$$\gamma(S, T) := \sum_{t \in T} \deg_G(t) - a|T| + b|S| - e_G(S, T)$$
$$= e(T, V(G)/S) + b|S| - a|T| \geq 0$$

for all disjoint subsets S and T of $V(G)$.

Lemma 2 *Let G be a graph, and a and b be integers such that $2 \le a < b$. Suppose that there exists disjoint subsets S and T of $V(G)$ satisfying $\gamma(S,T) < 0$. Among all such subsets, choose subsets S and T so that $|T|$ is minimum. Then following statements hold.*
(i) $T \neq \emptyset$.
(ii) $e(t, V(G)/S) \le a-1$ *for all* $t \in T$. *(1)*
(iii) If $a = 2$, *then* $|\Gamma(T)| < (1+\frac{1}{b})|T|$. *(2)*

Proof. (i) Since $\gamma(S, \emptyset) = b|S| \ge 0$, we have $T \neq \emptyset$.

(ii) Assume that there exists $t \in T$ such that $e(t, V(G)/S) \ge a$. Then $\gamma(S, T/t) = \gamma(S,T) - e(t, V(G)/S) + a \le \gamma(S,T) < 0$, which contradicts the choice of T. Hence (ii) holds.

(iii) Assume $a = 2$. Then we have

$$\begin{aligned} |\Gamma(T)| &\le |S| + e(T, V(G)/S) \\ &< \frac{2}{b} + (1 - \frac{1}{b})e(T, V(G)/S) \quad \text{(by } \gamma(S,T) < 0) \\ &\le \frac{2}{b} + (1 - \frac{1}{b})|T| \qquad \text{(by (1) and } a = 2) \\ &= (1 + \frac{1}{b})|T|. \end{aligned}$$

Lemma 3 *Lemma G be a graph of order n which satisfies the assumption of Theorem 2. Then* $\delta(G) \ge \frac{\lambda n+1}{1+\lambda}$.

Proof. Let v be a vertex of G with degree $\delta(G)$. Put $X = V(G)/\Gamma(v)$. Since $\Gamma(X)$ does not contain v, we have $(1+\lambda)|X| \le |\Gamma(X)| \le n-1$. Hence $(1+\lambda)(n - \delta(G)) \le n-1$, and thus $\delta(G) \ge (\lambda n + 1)/((1+\lambda)$.

Proof of Theorem 2. Suppose that a graph G of order n satisfies the assumption of the theorem, but has no $[a,b]$-factor. By Lemma 1, there exists disjoint subsets S and T of $V(G)$ such that $\gamma(S,T) < 0$. Among all such subsets, choose subsets S and T so that $|T|$ is minimum. Define an integer μ by

$$\mu := \min_{t \in T} e(t, V(G)/S).$$

Then $0 \le \mu \le a-1$ by (1). We derive a contradiction by considering three cases.

Case 1. $(1+\lambda)|T| \le |\Gamma(T)|$ and $\mu \neq 1$.

By (2), we have $a \le 3$, Since $(1+\lambda)|T| \le |\Gamma(T)| \le |V(G)| = n$, we have

$$|T| \le \frac{n}{1+\lambda}. \tag{3}$$

By the definition of μ and Lemma 3, we have

$$|S| \geq \delta(G) - \mu \geq \frac{\lambda n + 1}{1+\lambda} - \mu. \tag{4}$$

We first assume $\mu \geq 2$. Then

$$\begin{aligned}\gamma(S,T) &= e(T, V(G)/S) + b|S| - a|T| \\ &\geq \mu|T| + b|S| - a|T| \\ &\geq b(\frac{\lambda n+1}{1+\lambda} - \mu) - \frac{(a-\mu)n}{1+\lambda}. \quad \text{(by (3) and (4))}\end{aligned}$$

The last expression is non-negative if and only if

$$n \geq \frac{b((1+\lambda)\mu - 1)}{b\lambda - a + \mu} = b(1+\lambda) + \frac{a-1}{\mu-1}. \quad (\text{by } \lambda = (a-1)/b)$$

Since $b(1+\lambda) + (a-1)/(\mu-1) \leq b + a - 1 + a - 1 < b + 2a < n$, we can conclude that $\gamma(S,T) \geq 0$, and get a contradiction.

We next assume $\mu = 0$. Put $k := \#\{t \in T | e(t, V(G)/S) = 0\} \geq 1$. It follows from Lemma 3 and $(1+\lambda)|T| \leq |\Gamma(T)| \leq n - k$ that $|S| \geq \delta(G) \geq (\lambda n + 1)/(1+\lambda)$ and $|T| \leq (n-k)/(1+\lambda)$. Hence we obtain

$$\begin{aligned}\gamma(S,T) &= e(T, V(G)/S) + b|S| - a|T| \\ &\geq |T| - k + b|S| - a|T| \\ &\geq \frac{b(\lambda n+1)}{1+\lambda} - \frac{(a-1)(n-k)}{1+\lambda} - k \\ &= \frac{(b\lambda - a + 1)n + b + (a - 2 - \lambda k)}{1+\lambda} \\ &= \frac{b + (a-2-\lambda)k}{1+\lambda} \geq 0 \quad (\text{by } a \geq 3 \text{ and } \lambda < 1)\end{aligned}$$

Therefore we have a contradiction.

Case 2. $(1+\lambda)|T| \leq |\Gamma(T)|$ and $\mu = 1$.

Put $h := \#\{t \in T | e(t, V(G)/S) = 1\} \geq 1$. By $\mu = 1$ and Lemma 3, we obtain

$$|S| \geq \delta(G) - 1 \geq \frac{\lambda n + 1}{1+\lambda} - 1 = \frac{\lambda(n-1)}{1+\lambda}.$$

Suppose $|T| \le (n-1)/(1+\lambda)$. Then we have

$$\begin{aligned}\gamma(S,T) &= e(T,V(G)/S) + b|S| - a|T| \\ &\ge 2|T| - h + b|S| - a|T| \\ &\ge \frac{b(\lambda n+1)}{1+\lambda} - \frac{(a-1)(n-k)}{1+\lambda} - h \\ &= \frac{n-1}{1+\lambda} - h \ge |T| - h \ge 0.\end{aligned}$$

This is a contradiction. Hence we may assume $|T| > (n-1)/(1+\lambda)$. Then $|S|+|T| > \lambda(n-1)/(1+\lambda)+(n-1)/(1+\lambda) = n-1$. So we have $|S|+|T| = n$. Since $(1+\lambda)|T| \le |\Gamma(T)| \le n$, we have $|T| \le n/(1+\lambda)$. Therefore

$$\begin{aligned}\gamma(S,T) &= e(T,V(G)/S) + b|S| - a|T| \\ &\ge |T| + b(n-|T|) - a|T| \quad (\text{by } \ \mu = 1) \\ &= bn - (b+a-1)|T| \\ &\ge bn - \frac{(b+a-1)n}{1+\lambda} = 0.\end{aligned}$$

Consequently $\gamma(S,T) \ge 0$, a contradiction.

Case 3. $(1+\lambda)|T| > |\Gamma(T)|$.

In this case, it follows that $\Gamma(T) = V(G)$. In particular, $1 \le \mu \le a-1$. Since $(1+\lambda)|T| > |\Gamma(T)| = n$, we have $|T| > n/(1+\lambda)$, and so $|T| \ge \lfloor n/(1+\lambda) \rfloor + 1$. If $\mu = 1$, then take a vertex $v \in T$ such that $e(v,V(G)/S) = \mu = 1$. Then $v \notin \Gamma(T/\Gamma(v))$. On the other hand, the inequality $|T/\Gamma(T)| \ge |T| - 1 \ge \lfloor n/(1+\lambda) \rfloor$ implies that $\Gamma(T/\Gamma(v)) = V(G)$, a contradiction. Hence we may assume that $\mu \ge 2$. We shall show that $|T| - \mu \ge n/(1+\lambda)$, by which we can get a contradiction as follows. Take a vertex v of T with $e(v,V(G)/S) = \mu$. Then $v \notin \Gamma(T/\Gamma(v))$. On the other hand, the inequality $|T/\Gamma(v)| \ge |T| - \mu \ge n/(1+\lambda)$ implies that $\Gamma(T/\Gamma(v) = V(G)$. Thus we can obtain a contradiction.

We now prove that $|T| - \mu \ge n/(1+\lambda)$. Since $0 > \gamma(S,T) = e(T,V(G)/S) + b|S| - a|T| \ge \mu|T| + b|S| - a|T|$, we have $|T| \ge b|S|/(a-\mu)$. Since $|S| \ge \delta(G) - \mu \ge (\lambda n+1)/(1+\lambda) - \mu$, we obtain

$$|T| - \mu \ge \frac{b|S|}{a-\mu} - \mu \ge \frac{b}{a-\mu}(\frac{\lambda n+1}{1+\lambda} - \mu) - \mu.$$

Hence it suffices to show that

$$\frac{b}{a-\mu}(\frac{\lambda n+1}{1+\lambda} - \mu) - \mu \ge \frac{n}{1+\lambda}.$$

This inequality is equivalent to the following inequality:

$$
\begin{aligned}
n \geq & \frac{(a-\mu+b)(1+\lambda)\mu-b}{b\lambda-a+\mu} = \frac{-(1+\lambda)\mu^2+(a+b)(1+\lambda)\mu-b}{\mu-1} \\
= & -(1+\lambda)\mu+(a+b-1)(1+\lambda)+\frac{(a+b-1)(1+\lambda)-b}{\mu-1} := f(\mu)
\end{aligned}
$$

Since $2 \leq \mu \leq a-1$, the function $f(\mu)$ attains its maximum at $\mu = 2$. Consequently, we have

$$
\begin{aligned}
f(\mu) \leq f(2) &= (2a+2b-4)(1+\lambda)-b \\
&= 4a+b+2a\lambda-6-4\lambda < 6a+b. \quad (\text{by } \lambda = (a-1)/b)
\end{aligned}
$$

Since $n \geq 6a+b$, we conclude that $|T|-\mu \geq n/(1+\lambda)$. Therefore, the proof of the theorem is complete.

Theorem 3 *Let a and b be integers such that either $5 \leq a < b$, or $2 \leq a < b$ and b is odd. Then there exists infinitely many graphs G which satisfy the following two conditions.*
(i) G has no $[a,b]$-factor.
(ii) For all subset X of $V(G)$, we have

$$
\Gamma(X) = V(G) \quad \textit{or} \quad |\Gamma(X)| \geq (1+\frac{a-1}{b})|X|.
$$

Proof We first assume that b is odd. Let m be any odd positive integer. We construct a graph G an follows. Let $V(G) = S \cup T$(disjoint union), $|S| = (a-1)m$ and $|T| = bm+1$, and put $T = \{t_1, t_2, \ldots, t_{2l}\}$, where $2l = bm+1$. For every vertex $s \in S$, define $\Gamma(s) = V(G)/\{s\}$, and for every $t \in T$, define $\Gamma(t) = S \cup \{t^{'}\}$, where $\{t, t^{'}\} == \{t_{2i-1}, t_{2i}\}$ for some i, $1 \leq i \leq l$. We first show that G has no $[a,b]$-factor.

$$
\begin{aligned}
\gamma(S,T) =& e(T, V(G)/S) + b|S| - a|T| \\
=& |T| + b|S| - a|T| \\
=& b(a-1)m - (a-1)(bm+1) = -a+1 < 0.
\end{aligned}
$$

Hence G has no $[a,b]$-factor by Lemma 1. We next show that the condition (ii) holds. Put $\lambda = (a-1)/b$. Let X be any vertex subset of G. It is obvious that if $|X \cap S| \geq 2$, or $|X \cap S| = 1$ and $|X \cap T| \geq 1$, then $\Gamma(X) = V(G)$. Of course, if $|X| = 1$ and $X \subseteq S$, then $|\Gamma(X)| = |V(G)|-1 \geq 1+\lambda$. Hence we may assume $X \subseteq T$. Since $|\Gamma(X)| = |S|+|X| = (a-1)m+|X|$,@$|\Gamma(X)| \geq (1+\lambda)|X|$ holds if and only if $(a-1)m+|X| \geq (1+\lambda)|X|$. This inequivalent to $|X| \leq bm$.

Thus if $X \neq T$ and $X \subset T$, then (ii) holds. If $X = T$, then $\Gamma(X) = V(G)$. Consequently, (ii) follows.

We now consider the case that $5 \leq a < b$ and b is even. Let m be any positive integer. We construct a graph G as follows. Let $V(G) = S \cup T$(disjoint union), $|S| = (a-1)m$ and $|T| = bm + 1$. Put $T = \{u, v, w\} \cup \{t_1, \ldots, t_{2l}\}$, where $2l = bm - 2$. For every $s \in S$, set $\Gamma(s) = V(G)/\{s\}$. For every $x \in \{u, v, w\}$, set $\Gamma(x) = S \cup (\{u, v, w\}/\{x\})$, and for every $t \in T/\{u, v, w\}$, let $\Gamma(t) = S \cup \{t'\}$, where $\{t, t'\} = \{t_{2i-1}, t_{2i}\}$ for some i, $1 \leq i \leq l$. Then $\gamma(S, T) = b|S| - (a-1)|T| + 3 = -a + 4 < 0$. Thus G has no $[a, b]$-factor. We can similarly show that the condition (ii) holds. Therefore the proof is complete.

It seems to be true that even if $2 \leq a \leq 4$ and b is even, there exists infinitely many graphs which satisfy the two conditions in Theorem 3. However it is not easy to find such graphs.

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