

# On finite homogeneous symmetric sets

Mikio Kano, Hiroshi Nagao and Nobuo Nobusawa

## 1 Introduction

A symmetric set is a set  $A$  on which a binary operation  $a \circ b$  is defined satisfying the following three axioms:

- (1.1)  $a \circ a = a$ ,
- (1.2)  $(x \circ a) \circ a = x$ ,
- (1.3)  $x \circ (a \circ b) = ((x \circ b) \circ a) \circ b$ .

The mapping  $S_a : A \rightarrow A$  defined by  $xS_a = x \circ a$  is a permutation on  $A$  by (1.2), and it is called the symmetry around  $a$ . Corresponding to the axioms above we have the following:

- (1.1')  $aS_a = a$ ,
- (1.2')  $S_a^2 = I$ ,
- (1.3')  $S_{a \circ b} = S_a S_b = S_b^{-1} S_a S_b$ .

We denote by  $G(A)$  the permutation group on  $A$  generated by  $S_A = \{S_a | a \in A\}$ . Since  $T^{-1}S_a T = S_{aT}$  for  $a \in A$  and  $T \in G(A)$  by (1.3'),  $S_A$  is a set of involutions in  $G(A)$  which is  $G(A)$ -invariant. The subgroup of  $G(A)$  generated by  $\{S_a S_b | a, b \in A\}$  is called the group of displacements and is denoted by  $H(A)$ . The set  $S_A$  is a symmetric set with binary operation  $S_a \circ S_b = S_b^{-1} S_a S_b$ . The mapping  $a \mapsto S_a$  of  $A$  onto  $S_A$  is a homomorphism, and if it is an isomorphism, i.e. if  $a \neq b$  implies  $S_a \neq S_b$  then  $A$  is called *effective*. If  $A$  is effective then the center  $Z(G(A))$  of  $G(A)$  is trivial.

**Remark** In [4] and [5] the group of displacements is denoted by  $G(A)$ .

Now suppose that  $G$  is a group and  $A$  is a subset of  $G$  satisfying the following:

- (1.4)  $A$  is a set of involutions in  $G$  which is  $G$ -invariant,
- (1.5)  $G$  is generated by  $A$ .

Then  $A$  is a symmetric set with the binary operation  $a \circ b = b_{-}ab$ , and it is easy to show that  $G(A)$  is isomorphic to  $G/Z(G)$ . If a symmetric set  $A'$  is isomorphic to  $A$  then we say that  $A'$  is embedded in a group  $G$ . The subgraph generated by  $H$ . If  $Z(G) = 1$  then  $A$  is said to be embedded *faithfully* in  $G$ . Every effective symmetric set  $A$  is embedded faithfully in the group  $G(A)$ .

A symmetric set  $A$  is called *homogeneous* if it satisfies the following condition:

(1.6)  $a \circ b = b$  has a solution  $x$  in  $A$  for any  $a, b \in A$ .

If  $A$  is homogeneous then  $S_A$  is a conjugate class of involutions in  $G(A)$ , and mapping  $\phi_a : x \mapsto a \circ x$  of  $A$  to  $A$  is surjective. Now suppose  $A$  is finite. Then  $\phi_a$  is also injective, and hence the solution  $x$  of  $a \circ b = b$  is unique. Especially  $aS_x \neq aS_y$  if  $x \neq y$ . Thus a finite homogeneous symmetric set  $A$  is effective and can be embedded faithfully in a finite group  $G$ . Then the condition (1.6) is equivalent to the following:

(1.7) for any  $a, b \in A$  there is  $c \in A$  such that  $c^{-1}ac = b$ .

In this way every finite homogeneous symmetric set  $A$  can be regarded as a conjugate class of involutions in a finite group  $G$  satisfying (1.5) and (1.7).

The purpose of this paper is to study the structure of finite homogeneous symmetric sets in connection with finite groups generated by a conjugate class of involutions satisfying (1.7). The following theorem, which will be proved in the next section by using the Glauberman's  $Z^*$ -Theorem, is fundamental.

**Theorem 1** *Suppose a finite symmetric set  $A$  is embedded in a group  $G$ . Then  $A$  is homogeneous if and only if the group of displacements  $H$  is of odd order.*

All sets considered in this paper are assumed to be finite. For a set  $X$ ,  $|X|$  denotes the cardinality of  $X$  and  $|X|_p$  denotes the  $p$ -part of  $|X|$  for a prime  $p$ . For a group  $G$ ,  $O(G)$  denote the maximal normal subgroup of  $G$  of odd order, and  $Z^*(G)$  is the subgroup containing  $O(G)$  such that  $Z^*(G)/O(G)$  coincides with the center of  $G/O(G)$ . For  $a \in G$ , the order of  $a$  is denoted by  $o(a)$ . When  $G$  acts on a set  $X$  the action is called *semiregular* if any  $a \neq 1$  of  $G$  has no fixed point. Other notation in group theory is the same as in [3].

## 2 Proof of Theorem 1 and preliminary lemmas

We begin with the following lemma.

**Lemma 1** *Let  $a$  and  $b$  be two involutions in a group  $G$ . Then the subgroup  $\langle a, b \rangle$  generated by  $a$  and  $b$  is the dihedral group of order  $2r$ , where  $r$  is the order of  $ab$ . If  $r = o(ab)$  is odd then  $\langle a, b \rangle = a\langle ab \rangle$  is a conjugate class of involutions in  $\langle a, b \rangle$  satisfying (1.7).*

**Proof.** Let  $x = ab$ . Then  $\langle a, b \rangle = \langle a, x \rangle$  and we have  $a^2 = 1, x^r = 1, a^{-1}xa = x^{-1}$ . Thus  $\langle a, b \rangle$  is the dihedral group of order  $2r$ . If  $r$  is odd, then since  $x^{-i}ax^i = ax^{2i}$  we have  $\{x^{-i}ax^i | 0 \leq i < r\} = a\langle x \rangle = \langle a, b \rangle - \langle ab \rangle$ . Hence  $a\langle x \rangle$  is the conjugate class in  $\langle a, b \rangle$  containing  $a$ , and for any element  $c$  of  $a\langle x \rangle$  there is an integer  $i$  such that  $c = ax^{2i}$ . Then  $c = (ax^i)^{-1}a(ax^i)$ . Since  $ax^i \in a\langle x \rangle$ ,  $a\langle x \rangle$  satisfies (1.7).

From now on we assume that  $A$  is a symmetric set which is embedded in a group  $G$ .

**Remark.** For  $e, a \in A$ , the cycle generated by  $a$  with a base point  $e$  which is defined in [5] coincides with the following sequence of element of  $A$ :  $e, a = e(ea), e(ea)^2, e(ea)^3, \dots$ .

Now suppose  $H = \langle ab | a, b \in A \rangle$  is of odd order. Then by Lemma 11  $A$  satisfies (1.7) and hence  $A$  is homogeneous. Thus the ‘if’ part of Theorem 1 is proved.

To prove the ‘only if’ part, we assume that  $A$  satisfies (1.7).

**Lemma 2** *Under the assumption above we have the following:*

- (i) For  $a, b \in A$  the element  $c$  of  $A$  satisfying  $c^{-1}ac = b$  is unique.
- (ii) For  $a \in A$ ,  $A \cap C_G(a) = \{a\}$ .
- (iii)  $|A|$  is odd.
- (iv) If  $a, b \in A$  then  $o(ab)$  is odd,  $\langle ab \rangle$  acts on  $A$  semi-regularly and hence  $o(ab)$  divides  $|A|$ .
- (v) For a fixed  $e \in A$ ,  $H = \langle ea | a \in A \rangle = G'$ .
- (vi)  $H$  is of odd order.

**Proof.** (i) By (1.7) the mapping  $x \mapsto x^{-1}ax$  of  $A$  to  $A$  is surjective, and hence injective.

(ii) Since  $a^{-1}aa = a$ , the assertion follows from (i).

(iii) For  $a \in A$ , the group  $\langle a \rangle$  of order 2 acts on  $A$  and it fixes only  $a$ . Hence  $|A|$  is odd.

(iv) Let  $D = \langle a, b \rangle$ . Then  $\langle a \rangle$  acts on  $a^D = \{d^{-1}ad | d \in D\}$ , and since  $a$  fixes only  $a$  in  $a^D$ ,  $|a^D|$  is odd. On the other hand  $\langle b \rangle$  also acts on  $a^D$ , and since  $|a^D|$  is odd  $b$  fixes an element  $y$  of  $a^D$ . Then by (ii)  $y = b \in a^D$ . Hence  $b = (ab)^{-i}a(ab)^i = a(ab)^{2i}$  for some  $i$ . Thus  $(ab)^{2i-1} = 1$  and hence  $o(ab)$  is odd.

Now suppose  $(ab)^{-i}(ab)^i = c$  for some  $c \in A$ . Then  $a^{-1}ca = [(ab)^i a]^{-1}c[(ab)^i a]$ . Since  $(ab)^i a \in A$  by Lemma 1, we have  $a = (ab)^i a$  by (i) and hence  $(ab)^i = 1$ . Thus if  $(ab)^i \neq 1$  then  $(ab)^i$  has no fixed element in  $A$ .

(v) For  $a, b \in A$ ,  $ab = (ea)^{-1}(eb)$ . Hence  $H = \langle ab | a, b \in A \rangle = \langle ea | a \in A \rangle$ . Since  $G = \langle A \rangle = H \cup eH$ ,  $|G : H| \leq 2$  and  $G' \leq H$ . On the other

hand for  $a \in A$  there is an element  $b$  of  $A$  such that  $a = b^{-1}eb$ , and then  $ea = e^{-1}b^{-1}eb \in G'$ . Hence  $G' = H$ .

(vi) Let  $a$  be an element of  $A$ . Then by (iv), for any  $g \in G$ ,  $f^{-1}a^{-1}ga$  is of odd order. Then by the Glauberman's  $Z^*$ -Theorem ([2], Theorem 1) we have  $a \in Z^*(G)$ . Since  $G = \langle A \rangle$ ,  $G = Z^*(G)$  and hence  $O(G) \geq G' = H$ .

The 'only if' part of Theorem 1 is proved in (vi) of Lemma 2. Now since  $G$  is of even order we have  $|G : H| = 2$ . By the Feit-Thompson's theorem  $G$  is solvable and by the Sylow's theorem all involutions are conjugate. Thus we have the following

**Corollary.** *If a homogeneous symmetric set  $A$  is embedded in a group  $G$ , then  $G$  is solvable,  $|G : H| = 2$ ,  $|H|$  is odd and  $A$  is the only conjugate class of involutions in  $G$ .*

Let  $e$  be a fixed element of  $A$ . Then  $e$  induces an involutive automorphism of the group  $H$  of odd order. Let  $V(e) = C_H(e)$  and  $K(e) = \{k \in A | e^{-1}ke = k^{-1}\}$ . Then we have the following

**Lemma 3** (i) *Each coset of  $V(e)$  in  $H$  contains only one element of  $K(e)$ , and hence  $|H : V(e)| = |K(e)|$ .*

(ii)  *$K(e) = \{ea | a \in A\}$ ,  $|A| = |K(e)|$  and  $H = \langle K(e) \rangle$ .*

(iii) *If a prime  $p$  divides  $|H|$  then  $p$  also divides  $|A|$ . In particular if  $|A|$  is a power of a prime  $p$  then  $H$  is a  $p$ -group.*

(iv) *Any  $e$ -invariant  $p$ -subgroup of  $H$  is contained in an  $e$ -invariant Sylow  $p$ -subgroup of  $H$ . If  $P$  is an  $e$ -invariant Sylow  $p$ -subgroup of  $H$ , then*

$$|A|_p = |K(e)|_p = |P \cap K(e)|, \quad |V(e)|_p = |P \cap V(e)|.$$

(v)  *$H$  is abelian if and only if  $H = K(e)$ .*

**Proof.** For the proofs of (i) and (v) see Lemma 2 in [1].

(ii) Since  $A$  is the conjugate class in  $G$  containing  $e$ , we have

$$|A| = |G : C_G(e)| = |H : C_H(e)| = |H : V(e)| = |K(e)|.$$

Now evidently  $\{ea | a \in A\} \subseteq K(e)$ . Hence we have  $K(e) = \{ea | a \in A\}$  and  $H = \langle K(e) \rangle$ .

(iii) If  $p$  does not divide  $|A| = |K(e)|$ , then a Sylow  $p$ -subgroup of  $V(e)$  is a Sylow  $p$ -subgroup of  $H$ . Then by (v) of Lemma 2.1 in [1]  $p$  does not divide  $|\langle K(e) \rangle| = |H|$ , which is a contradiction.

(v) If  $H$  is abelian then  $K(e)$  is a subgroup, and hence  $H = K(e)$ . Conversely suppose that  $H = K(e)$ . Then for  $x, y \in H$   $(xy)^{-1} = (xy)^e = x^e y^e = x^{-1} y^{-1}$ . Hence  $x$  and  $y$  commute and  $H$  is abelian.

### 3 Symmetric sets which are also groups

Let  $X$  be any group. Then defining the binary operation on  $X$  by setting  $x \circ y = yx^{-1}y$   $X$  is a symmetric set. In this case we say that the symmetric set  $X$  is also a group.

**Theorem 2** *Let  $A$  be a symmetric set which is also a group. Then  $A$  is homogeneous if and only if  $A$  is of odd order.*

**proof** If  $A$  is homogeneous then by (iii) of Lemma 2  $|A|$  is odd.

Conversely suppose that  $A$  is a group of odd order. It suffices to show that the mapping  $x \mapsto a \circ x = xa^{-1}x$  is injective, and hence surjective. Since  $A$  is of odd order the mapping  $x \mapsto x^2$  of  $A$  to  $A$  is bijective. Let  $a = b^2$  and assume  $xb^{-1}x = yb^{-2}y$ . Then we have  $(bx^{-1}b)^2 = (by^{-1}b)^2$ ,  $bx^{-1}b = by^{-1}b$  and hence  $x = y$ .

The following is obtained in [5]. For the completeness we shall prove it in a slightly different way.

**Theorem 3** *Let  $A$  be an effective symmetric set. Then the following conditions are equivalent:*

- (i)  $A$  is also an abelian group.
- (ii) The group of displacements  $H(A)$  is abelian.
- (iii)  $H(A) = \{S_e S_a | a \in A\}$ , where  $e$  is a fixed element of  $A$ .

*Furthermore if one of the conditions is satisfied the  $A$  is homogeneous and hence  $|A|$  is odd.*

**Proof.** (i)  $\Rightarrow$  (ii) suppose that  $A$  is also an abelian group. Then  $a \circ b = ba^{-1}b = a^{-1}b^2$ . Since  $xS_e S_a = xe^{-2}a^2$ ,  $S_e S_a$  and  $S_e S_b$  commute. Hence  $H(A) = \langle S_e S_a | a \in A \rangle$  is abelian.

(ii)  $\Rightarrow$  (iii) Let  $e$  be a fixed element of  $A$ . Then, since  $H(A)$  is abelian and  $S_e$  inverts  $S_e S_a$ ,  $S_e$  inverts every element of  $H(A)$ . Suppose  $H(A)$  has an involution  $T$ . Then  $T$  commutes with  $S_e$ , hence  $T$  is in the center  $Z(G(A))$  of  $G(A)$ , which is a contradiction. Thus  $H(A)$  is of odd order, and by Theorem 1  $A$  is homogeneous. By (v) of Lemma 3 we have  $H(A) = \{S_e S_a | a \in A\}$ .

(iii)  $\Rightarrow$  (i) Suppose  $H(A) = \{S_e S_a | a \in A\}$ . Since  $S_e$  inverts every element of  $H(A)$ ,  $H(A)$  is an abelian group. Then it is easy to see that the mapping  $a \mapsto S_e S_a$  of  $A$  onto  $H(A)$  is an isomorphism of symmetric sets. Thus  $A$  is also an abelian group.

The last half of the theorem has been shown in the proof of (ii)  $\Rightarrow$  (iii).

A symmetric set  $A$  is called abelian if  $H(A)$  is abelian group.

## 4 Symmetric subsets

Let  $A$  be a symmetric set. A subset  $B$  of  $A$  is called a *symmetric subset* of  $A$  if  $b \circ c \in B$  for any  $b, c \in B$ . If  $B$  is a symmetric subset of  $A$  then  $B \circ a$  is also a symmetric subset, and if  $A$  is homogeneous then  $B$  is also homogeneous and  $B \cap B \circ a = \emptyset$  for  $a \in A - B$ .

From now on we assume that  $A$  is a homogeneous symmetric set which is embedded in a group  $G$ , and let  $H = \langle ab | a, b \in A \rangle$ . If  $B$  is a symmetric subset of  $A$  then  $B$  is embedded in  $G_B = \langle B \rangle$ . Let  $H_B = \langle bc | b, c \in B \rangle$ .

**Theorem 4** (i) *Let  $B$  be a subset of  $A$  and  $e \in B$ . Then  $B$  is a symmetric subset if and only if there exists an  $e$ -invariant subgroup  $J$  of  $H$  such that  $B = e^J = \{j^{-1}ej | j \in J\}$ .*

(ii) *A symmetric subset  $B$  is abelian if and only if there exists an  $e$ -invariant abelian subgroup  $J$  of  $H$  such that  $B = e^J$ .*

**Proof.** If  $B$  is a symmetric subset of  $A$ , then  $H_B = \langle eb | b \in B \rangle$  is  $e$ -invariant and  $B = e^{H_B}$ . By Theorem 3  $B$  is abelian if and only if  $H_B$  is abelian. Suppose conversely that  $J$  is an  $e$ -invariant subgraph of  $H$  and  $B = e^J$ . Then for  $j, k \in J$ ,  $e^j \circ e^k = e^{-k}e^je^k = k^{-1}(jk^{-1})^{-e}e(jk^{-1})^ek \in e^J$ . Hence  $B$  is a symmetric subset.

**Theorem 5** *If  $B$  is a symmetric subset of a homogeneous symmetric set  $A$ , then  $|B|$  divides  $|A|$ .*

**Proof.** Let  $e \in B$  and  $p$  a prime division of  $|B|$ . By (iv) of Lemma 3 there is an  $e$ -invariant Sylow  $p$ -subgroup  $Q$  of  $H_B$  and  $Q$  is contained in an  $e$ -invariant  $p$ -subgroup  $P$  of  $H$ . Then

$$|A|_p = |P \cap K(e)|_p \geq |Q \cap K(e)|_p = |B|_p.$$

Hence  $|B|$  divides  $|A|$ .

A symmetric subset  $B$  of  $A$  is called a symmetric  $p$ -subset if  $|B|$  is a power of  $p$ , and  $B$  is called a symmetric Sylow  $p$ -subset if  $|B| = |A|_p$ . Then we have the following Sylow's theorem for homogeneous symmetric sets.

**Theorem 6** *Let  $C$  be a symmetric  $p$ -subset of a homogeneous symmetric set  $A$ . Then  $C$  is contained in a symmetric Sylow  $p$ -subset of  $A$ . Two symmetric Sylow  $p$ -subsets of  $A$  are isomorphic.*

**Proof.** Let  $e \in C$ . By (iii) of Lemma ??  $H_C$  is an  $e$ -invariant  $p$ -subgroup of  $H$  and is contained in an  $e$ -invariant Sylow  $p$ -subgroup  $P$  of  $H$ . Let  $B = e^P$ . Then  $C = e^{H_C} \subseteq B$ , and since

$$|B| = |P : P \cap V(e)| = |P \cap K(e)| = |A|_p$$

$B$  is a symmetric Sylow  $p$ -subset of  $A$ .

## 5 Symmetric quotient sets

Suppose that an equivalence relation  $\sim$  in a symmetric set  $A$  satisfies the following condition: if  $a \sim a'$  and  $b \sim b'$  then  $a \circ b \sim a' \circ b'$ . Denote the equivalence class containing  $a$  by  $a^*$ . Then the set of all equivalence classes  $A^* = A/\sim$  is a symmetric set with the binary operation  $a^* \circ b^* = (a \circ b)^*$ . We call  $A^*$  a symmetric quotient set of  $A$  and an equivalence class is called a coset. Since  $b \circ c \sim a \circ a = a$  for  $b, c \in a^*$ , each coset is a symmetric subset of  $A$ .

Now suppose  $A$  is homogeneous. Then a symmetric quotient set  $A^*$  of  $A$  is also homogeneous. Let  $e \in A$  and  $B = e^*$ . If  $x \sim e \circ a$  then  $x \circ a \sim (e \circ a) \circ a = e$  and hence  $x = (x \circ a) \circ a \in B \circ a$ . Thus  $(e \circ a)^* \subseteq B \circ a$ . On the other hand if  $b \sim e$  then  $b \circ a \sim e \circ a$ . Hence  $B \circ a \subseteq (e \circ a)^*$  and we have  $(e \circ a)^* = B \circ a$ . Since  $A$  is homogeneous every coset can be written in a form  $B \circ a$  with  $a \in A$ . Therefore  $A^*$  is uniquely determined by a coset  $B$ , and hence we may denote  $A^*$  by  $A/B$ . A symmetric subset  $B$  of  $A$  is called *normal* in  $A$  if  $B$  is a coset of some symmetric quotient set of  $A$ .

Let  $A$  be a homogeneous symmetric set embedded in a group  $G$ ,  $H = \langle ab|a, b \in A \rangle$  and  $e \in A$ . If  $J$  is a subgroup of  $H$  which is normal in  $G$ , then  $\bar{A} = A \text{ mod } J$  is a symmetric set which is homogeneous to  $A$ , and  $\bar{A}$  is embedded in  $\bar{A} = G/J$ . Then the group of its displacements is  $\bar{A} = H/J$ .

**Theorem 7** (i) *Let  $B$  be a symmetric subset of  $A$  containing  $e$ . Then  $B$  is normal in  $A$  if and only if there exists a normal subgroup  $J$  of  $G$  such that  $J \subseteq H$  and  $B = e^J$ . In this case  $A/B$  is isomorphic to  $\bar{A} = A \text{ mod } J$ .*

(ii) *Let  $B$  be a symmetric normal subset of  $A$ . Then  $A/B$  is abelian if and only if there exists a normal subgroup  $J$  of  $G$  such that  $B = e^J$ ,  $J \subseteq H$  and  $H/J$  is abelian.*

**Proof** Suppose first that  $J$  is a normal subgroup of  $G$  contained in  $H$ . Let  $\bar{G} = G/J$  and  $\bar{A} = \{\bar{a} = aJ|a \in A\}$ . Let  $a^* = \{b \in A|\bar{b} = \bar{a}\}$ . Then  $A^* = \{a^*|a \in A\}$  is symmetric quotient set of  $A$  and  $A^* \simeq \bar{A}$ . Suppose  $\bar{a} = \bar{b}$  for  $a, b \in A$ . Then  $b = aj$  with  $j \in J$ , and since  $a$  and  $b$  are involutions  $a^{-1}ja = j^{-1}$ . Since  $J$  is of odd order there is an element  $i$  of  $J$  such that  $i^2 = j$ . Then  $a^{-1}ia = i^{-1}$  and we have  $b = i^{-1}ai \in a^J$ . Conversely if  $b \in a^J$  then  $\bar{a} = \bar{b}$ . Thus we have  $a^* = a^J$  and  $a^J$  is a coset. By Theorem 3  $\bar{A} (\simeq A^* = A/e^J)$  is abelian if and only if  $\bar{H} = H/J$  is an abelian group.

Suppose next that  $B = e^*$  is a coset of a symmetric quotient set  $A^*$  of  $A$ .

If  $a^* = b^*$  then for  $c \in A$   $(a^c)^* = (b^c)^*$ , and hence  $(a^x)^* = (b^x)^*$  for any  $x \in G$ . Since  $B^a = B^b$  we have  $B^{ab} = B$ . Let  $J = \langle ab|a, b \in A, a^* = b^* \rangle$ . Then  $J$  is a normal subgroup of  $G$  contained in  $H$  and  $e^J \subseteq B$ . Since  $H_B = \langle eb|b \in B \rangle \subseteq J$ , and  $B = e^{H_B}$ , we have  $e^J = B$ .

By using the solvability of  $H$ , we have the following

**Corollary 8** *If  $A$  is a homogeneous symmetric set, then there is a chain of symmetric subsets  $A = B_0 \supset B_1 \supset \cdots \supset B_n = \{e\}$  such that  $B_{i+1}$  is normal in  $B_i$  and  $B_i/B_{i+1}$  is abelian.*

Let  $Z$  be the center of  $H$ . Then  $Z$  is clearly a normal subgroup of  $G$  and hence by Theorem 7  $e^Z$  is a normal symmetric subsets of  $A$  which is abelian by (ii) of Theorem 4. In [4]  $e^Z$  is called the center of (relative to a base point  $e$ ). Now suppose that  $A$  is faithfully embedded in  $G$ . Then  $|e^Z| = |Z|$ . If  $A$  is a symmetric  $p$ -set then  $H$  is a  $p$ -group by (iii) of Lemma 3 and hence  $H$  has a non-trivial center. Thus we have

**Corollary 9** *If  $A$  is a homogeneous symmetric  $p$ -set, then the center of  $A$  relative to a base point  $e$  is not trivial.*

## References

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