ON FACTORS WITH GIVEN COMPONENTS

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Abstract

For a set $\{A, B, C, \cdots\}$ of graphs, an $\{A, B, C, \cdots\}$-factor of a graph G is defined to be a spanning subgraph of G each component of which is isomorphic to one of $\{A, B, C, \cdots\}$. The star with $n+1$ vertices is denoted by $K_{1,n}$, and let $\{T_n\}$ denote the set of trees with n vertices. We give a criterion for the existence in a graph of a $\{K_{1,1}, \cdots, K_{1,n}\}$-factor and that for the existence in a tree of $\{\{T_n\}\}$-factor.

1 Introduction

Consider a finite graphs G with vertex set $V(G)$ and edge set $E(G)$, which has neither multiple edges nor loops. We denote by $i(G)$ the number of isolated vertices of G. For a vertex subset S of G, we denote by $G - S$ the subgraph of G obtained from G by deleting the vertices in S together with their incident edges, and by $N(S)$ the *neighborhood* of S, that is , $N(S)$ is the set of vertices of G adjacent to at least one vertices in S. A vertex subset U of G is said to be *independent* if no two vertices in U are adjacent in G. The complete graph and the star with n vertices are denoted by K_n and $K_{1,n-1}$, respectively.

We now introduce a new notion concerning factors. For a set$\{A, B, C, \cdots\}$ of graphs, an$\{A, B, C, \cdots\}$-factor of a graph G is a spanning subgraph of G each component of which is isomorphic to one of$\{A, B, C, \cdots\}$. That is, if F is an$\{A, B, C, \cdots\}$ of G, then F is a subgraph of G such that $V(F) = V(G)$

and each component of F is contained in$\{A, B, C, \cdots\}$. For example, a 1-factor and a $\{K_2\}$-factor are the same, and we give two graphs in Fig. 1, one has a $\{K_{1,1}, K_{1,1}, K_{1,3}\}$-factor and the other has no $\{K_{1,1}, K_{1,1}, K_{1,3}\}$-factor.

A criterion for the existence of a $\{K_2\}$-factor was found by Tutte([6], [4,p.76]), and the following two proposition give criterions for existence of other factors.

(a) (b)

Fig.1. (a)A $\{K_{1,1}, K_{1,2}, K_{1,3}\}$-factor. (b)A graph having no $\{K_{1,1}, K_{1,2}, K_{1,3}\}$-factor.

Proposition 1 *(Tutte[7]; Berge and Hajnal [2,3,5]). For a geaph G , the following statements are equivalent:*

(1) G has a $\{K_2, \{C_n\}\}$-factor, where $\{C_n\}$ denotes the set of cycles with at least 3 vertices and $\{K_2, \{C_n\}\} = \{K_2, C_n | n \geq 3\}$.

(2) $i(G - S) \leq |S|$ for every $S \subset V(G)$.

(3) $|N(U)| \geq |U|$ for every independent $U \subset V(G)$.

Proposition 2 *(Akiyama, Avis and Era[1]). A graph G has a $\{K_{1,1}, K_{1,2}\}$-factor if and only if $i(G - S) \leq 2|S|$ for every $S \subset V(G)$*

Note that a graph G has $\{1, 2\}$-factor, which is a spanning subgraph of G such that the degree of each vertex is 1 or 2, if and only if G has a $\{K_{1,1}, K_{1,2}\}$-factor.

In this paper, we shall prove the following two theorems.

Theorem 1 *Let G be a graph and n be an integer greater than or equal to 2. Then, the following statements are equivalent:*

(1) G has a $\{K_{1,1}, K_{1,2}, \cdots K_{1,n}\}$-factor.

(2) $i(G - S) \leq n|S|$ for every $S \subset V(G)$.

(3) $|N(U)| \geq (1/n)|U|$ for every independent $U \subset V(G)$.

Theorem 2 *Let T be a tree and and $\{T_n\}$ be the set of trees with n vertices. For any vertex v of T, if $T - v$ has the components $C_1, C_2, \cdots, C_r$, then let p_i be the number such that $|V(C_i)| \equiv p_i (mod\ n)$ and $0 \leq p_i < n$ for every i. Then, a tree T has a $\{\{T_n\}\}$-factor if and only if $\sum p_i = n - 1$ for every vertex v of T.*

Note that if a tree has a $\{\{T_n\}\}$-factor, then T has exactly one $\{\{T_n\}\}$-factor, which will be proved in the proof of Theorem 2.

Theorem 1 is a generalized result of Proposition 2, and its proof is similar to that of Proposition 2. Theorem 2 can be considered as a generalization of the next result.

Proposition 3 *(Chungphaisan[4, p.80]). A tree T has a $\{K_2\}$-factor if and only if $T - v$ has exactly one odd component for every vertex v of T.*

2 Definition and notions

Let G be a graph. The edge of G joining two vertices v and w is denoted by vw or wv. A vertex sequence $v_1 v_2 \cdots v_n$ is called a *path* from v_1 to v_n of length $n-1$ if every $v_i v_{i+1} (1 \le i \le n-1)$ is an edge of G and all the vertices are distinct. If v is a vertex of a subgraph L of G, then $d_L(v)$ denotes the degree of v in L, that is, $d_L(v)$ is the number of edges of L incident with v. If S is a vertex subset of L, then $N_L(S)$ denote the neighborhood of S in L. For edges $ab \in E(L)$ and $cd \in E(G) - E(L)$, we denote by $L - ab$ and $L + cd$ the edge-induced subgraphs of G whose edge sets are $E(L) - \{ab\}$ and $E(L) \cup \{cd\}$, respectively. For a set $\{A, B, C, \cdots\}$ of a graphs, a subgraph M of G is said to be an $\{A, B, C, \cdots\}$-subgraph is each component of M is isomorphic to one of $\{A, B, C, \cdots\}$. Then, an $\{A, B, C, \cdots\}$-factor of G is a spanning $\{A, B, C, \cdots\}$-subgaraph of G. An $\{A, B, C, \cdots\}$-subgraph N of G is said to be *maximum* if G has no $\{A, B, C, \cdots\}$-subgraph $N^{'}$ such that $|V(N^{'})| > |V(N)|$(see Fig.2).

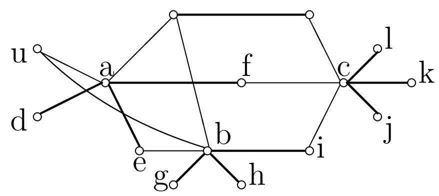

Fig.2. A maximum $\{K_{1,1}, K_{1,2}, K_{1,3}\}$-subgraph.

We now give some notations concerning alternating paths since we shall prove Theorem 1 using the standard technique of alternating paths. For a subgraph H of G an H-alternating path of G is a path whose edges are alternating in $E(H)$ and not in $E(H)$. For example, if H is the maximum $\{K_{1,1}, K_{1,2}, K_{1,3}\}$-subgraph shown in Fig.2, then paths $uaebic$ and $uafck$ are H-alternating ones. Let K be a subgraph of G and u be a vertex of G.

We denote by $A(u)$ the set of vertices w of G such that there exists a K-alternating path from u to w. Furthermore, we define two subsets $OA(u)$ and $EA(u)$ of $A(u)$ to be the sets of vertices w of G such that there exists a K-alternating path from u to w of odd length and of even length, respectively. For convenience, let $u \in A(u)$ and $u \notin EA(u)$.For example, if K is the maximum $\{K_{1,1}, K_{1,2}, K_{1,3}\}$-subgraph in Fig.2, then $EA(u) = \{d, e, f, g, i, j, k\}$ and $OA(u) = \{a, b, c\}$.

3 Proof of theorems

We first prove Theorem 1. In order to do so, we require the following lemma.

Lemma 1 *Let G be a graph having no $\{K_{1,1}, \cdots, K_{1,n}\}$-factors ($n \geq 2$) and let H be a maximum $\{K_{1,1}, \cdots, K_{1,n}\}$-subgraph of G. If u is a vertex of G not contained in H, then the following statements hold:*

(1) If $ux_1y_1x_2y_2 \cdots x_ry_r$ is an H-alternating path, then $d_H(x_i) = n$ and $d_H(y_i) = 1$ for every i.

(2) $A(u) - \{u\} \subseteq V(H)$, and $A(u)$ is a disjoint union of $\{u\}$, $OA(u)$ and $EA(u)$.

(3) $|EA(u)| = n|OA(u)|$.

(4) If a vertex w of G is adjacent in G to some vertex of $EA(u)$, then w is contained in $OA(u)$.

Proof. We first prove (1). If $d_H(x_1) = d_H(y_1) = 1$, then $H_1 = H + ux_1$ is a $\{K_{1,1}, \cdots, K_{1,n}\}$-subgraph with $V(H_1) = V(H) \cup \{u\}$, a contradiction. If $d_H(x_1) = 1$ and $d_H(y_2) \geq 2$, then $H_2 = H + ux_1 - x_1y_1$ is a $\{K_{1,1}, \cdots, K_{1,n}\}$-subgraph with $V(H_2) = V(H) \cup \{u\}$, a contradiction. If $2 \leq d_H(x_1) \leq n - 1$, then $d_H(y_1) = 1$ and $H_3 = H + ux_1$ is a $\{K_{1,1}, \cdots, K_{1,n}\}$-subgraph with $V(H_3) = V(H) \cup \{u\}$, a contradiction. Hence we have $d_H(x_1) = n$ and $d_H(y_1) = 1$. We can prove similarly one by one that $d_H(x_2) = n$, $d_H(y_2) = 1, \cdots, d_H(x_r) = n$, $d_H(y_r) = 1$. For example, if $2 \leq d_H(x_2) \leq n - 1$, then we obtain a contradiction considering $H + y_1x_2 - x_1y_1 + ux_1$.

We next prove (2). It is immediate that $A(u) - \{u\} \subseteq V(H)$. Then we have by (1) that

$$OA(u) = \{w \in A(u) | d_H(w) = n\} \quad \textit{and} \quad EA(u) = \{w \in A(u) | d_H(w) = 1\}.$$

Hence $OA(u) \cap EA(u) = \emptyset$ as $n \geq 2$. Therefore, (2) holds.

Statement (3) follows easily from the fact that, for each vertex w in $OA(u)$, there exist exactly n vertices in $EA(u)$ which are adjacent in H to w but not to any vertex in $OA(u) - \{w\}$.

Suppose that a vertex w of G is adjacent to a vertex v in $EA(u)$. Let $ux_1y_1 \cdots x_ry_r(y_r = v)$ be an H-alternating path of even length from u to v. Then $N_H(v) = \{x_r\}$. If w is not in the path, then $ux_1y_1 \cdots x_ry_r$ is an H-alternating path of odd length and so $w \in OA(u)$. If $w = x_j$ for some $j (1 \leq j \leq r)$, then $w \in OA(u)$ as $x_j \in OA(u)$. If $w = y_j$ for some j, then $ux_1y_1\cdots x_ry_r$ be an H-alternating path of odd length, and thus $v \in OA(u)$, which is contrary to (2). Consequently, (4) is proved.

Proof of Theorem 1. (1) $\Rightarrow$ (2). Suppose that G has a $\{K_{1,1},\cdots,K_{1,n}\}$-factor F. Let $C_1,\cdots,C_r$ be the components of F. Then, for any vertex subset S of G, we have

$$i(G-S) \le i(F-S) = \sum i(C_j - \{S \cap V(C)\}) \le \sum n|S \cap V(C_j)| = n|S|.$$

(2) $\Rightarrow$ (1). Suppose that G has no $\{K_{1,1},\cdots,K_{1,n}\}$-factors. Let H be a maximum $\{K_{1,1},\cdots,K_{1,n}\}$-subgraph of G and u be a vertex of G not contained in H. Then it follows from the lemma 1 that every vertex in $EA(u)$ is isolated in $G - OA(u)$, and it is obvious that u is isolated vertex of $G - OA(u)$. Hence we have

$$i(G = OA(u)) \ge |EA(u) \cup \{u\}| = n|OA(u)| + 1 > n|OA(u)|.$$

Consequently, the proof is complete.

(1) $\Rightarrow$ (3). Let F be a $\{K_{1,1},\cdots,K_{1,n}\}$-factor of G, and let $C_1,\cdots,C_r$ be the components of F. Then, for any independent vertex subset U of G, we have

$$|N_G(U)| \ge |N_F(U)| = \sum |N_{C_j}(V(C_j)) \cap U| \ge \sum (1/n)|V(C_j) \cap U| = |U|/n.$$

(3) $\Rightarrow$ (1). Suppose that G has no $\{K_{1,1},\cdots,K_{1,n}\}$-factors. Let H, u, $EA(u)$ and $OA(u)$ be the same as in the lemma 1. Then $EA(u)\cup\{u\}$ is an independent vertex subset of G, and we have

$$|N(EA(u) \cup \{u\})| = |OA(u)| = (1/n)|EA(u)| < (1/n)|EA(u) \cup \{u\}|.$$

Proof of Theorem 2. Suppose that a tree T has a $\{\{T_n\}\}$-factor F. Let v be any vertex of T and $C_1,\cdots,C_r$ be the components of $T-v$. Set $|V(C_i)| \equiv p_i (mod\ n)$, $0 \le p_i < n$. If v is contained in a component H of F, then

$$\sum p_i = \sum |V(C_i) \cap (V(H) - \{v\})| = |V(H) - \{v\}| = n - 1.$$

We next prove the sufficiency of the theorem by induction on the number of vertices of a tree T. It follows immediately that the number of vertices of T is divisible by n. If $|V(T)| = n$, then T has a $\{\{T_n\}\}$-factor as $T \in \{T_n\}$. Hence we may assume $|V(T)| \ge 2n$.

We first consider the case that there exists a vertex v such that $T-v$ has a component C with nm vertices for some positive integer m. It is easy to show that two trees C and $T-V(C)$ satisfy the condition in Theorem 2. Hence C and $T-V(C)$ have $\{\{T_n\}\}$-factor by the inductive hypothesis. Therefore, T has a $\{\{T_n\}\}$-factor.

We next consider the case that, for every vertex v of T, $T-v$ has no component with nm vertices. We shall prove that for any connected subgraph P of T, if $T-V(P)$ has the components $D_1,\cdots,D_r$ with $|V(D_i)| \equiv d_i(mod\ n)$, $0 \le d_i < n$, then

$$(*) \qquad |V(P)| + \sum d_i = n \ \ and\ \ d_i \ge 1 \ \ for\ every\ i.$$

We prove $(*)$ by induction on $|V(P)|$. We may assume $|V(P)| \ge 2$ since $(*)$ follows if $|V(P)| = 1$. Let u be an end-vertex of P and $V(P) = S \cup \{u\}$. Assume $T-S$ has the the components $X_1,\cdots,X_t$ with $|V(X_i)| \equiv x_i(mod\ n)$ and $u \in V(X_1)$. Then $\sum x_i + |S| = n$ and $x_i \ge n$ by the inductive hypothesis. Let the components of $T-u$ be $Y_1,\cdots,Y_s$ with

$$|V(Y_i)| \equiv y_i(mod\ n) \ \ and\ \ V(Y_1) \supseteq S \cup V(X_2) \cup \cdots \cup V(X_t).$$

Then $\sum y_i = n-1,\ y_i \geq 1$ and $y_1 = n - x_1$ as $x_1 + y_1 \equiv 0(mod\ n)$ and $x_1 \geq 1$. Since the set of components of $T - V(P)$ is $\{D_1, \cdots, D_r\} = \{X_2, \cdots, X_t, Y_2, \cdots, Y_s\}$, we obtain

$$|V(P)| + \sum d_i = 1 + |S| + \sum x_i - x_1 + \sum y_i - y_1 = 1 + n - x_1 - y_1 + \sum y_i = n.$$

Therefore $(*)$ follows. Hence $|V(T)| \leq n$, which is contrary to $|V(T)| \geq 2n$.

Moreover, we can show that a tree has a unique $\{\{T_n\}\}$-factor if it has one. Suppose a tree T has two distinct $\{\{T_n\}\}$-factors F and $F^{'}$. Then there exists an edge $e \in E(F) - E(F^{'})$, and let $T - e$ be the graph obtain from T by deleting e. Since $e \in E(F)$, the number of vertices of each component of $T - e$ is not divisible by n. On the other hand, it is divisible by n since $e \notin E(F^{'})$, a contradiction.

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References

[1] J.Akiyama, D.Avis and H.Era, on a $\{1,2\}$-factor, *TRU Math.* **16(2)** (1980) 97–102.

[2] C.Berge, Regularisable graph, in: *B. Bollobas, ed., Advances in Graph Theory, Annals Disctete Math.* **3** (North-Holland, Amsterdam, 1978) 11–19.

[3] C.Berge, Some common properties for regularizable graph graph, edge-critical graphs and **B**-graphs, in: *Graph Theorey and Algorithms, Lecture Notes in Computer Science* **108** (Springer-Verlag, Berlin, 1981).

[4] J.A.Bondy and U.S.R.Murty, Graph Theory with Aplications, *(Macmillan, London, 1976).*

[5] A.Hajnal, A theorem on k-saturated graphs, *Canad. J. Math.* **17** (1965) 720–772.

[6] W.T. Tutte, The fractorization of linear graphs, *J. London Math, Soc.* **22** (1947) 107–111.

[7] W.T. Tutte, The factor of graphs, *Canad. J. Math.* **4** (1952) 314–328.